**Math 242, Topics in Algebra – Spring 2023**
`https://sites.aub.edu.lb/kmakdisi/`
**Problem set 11, NOT DUE; use it to practice the last sections before the final**

**Exercises from Fraleigh:**
Section 55, exercises 1, 2, 6, 8, 10, 11, 12.
Section 56, exercises 1, 4, 5, 8.

**Additional Practice Exercises:**

**Exercise A11.1:** We consider the splitting field $E$ over $\mathbf{Q}$ of the polynomial $x^6 + 3$. Let $\alpha$ be a root of $x^6 + 3$, and let $\zeta$ be a primitive 6th root of unity. (It probably helps to view $\alpha$ and $\zeta$ as elements of $\mathbf{C}$).

a) Find $\mathrm{irr}(\alpha, \mathbf{Q})$ and $\mathrm{irr}(\zeta, \mathbf{Q})$. Justify, as always.

b) Show that $E = \mathbf{Q}(\alpha, \zeta)$. List explicitly the conjugates (over $\mathbf{Q}$) of each of $\alpha$ and $\zeta$, expressing the conjugates as elements in $\mathbf{Q}(\alpha, \zeta)$.

c) Show that in fact one can take $\zeta = (1 + \alpha^3)/2$, so that $E = \mathbf{Q}(\alpha)$.

d) List all the elements of $\mathrm{Gal}(E/\mathbf{Q})$, indicating their effect on each of $\alpha$ and $\zeta$.

e) Show that $\mathrm{Gal}(E/\mathbf{Q})$ is not abelian by finding two explicit elements $\sigma, \tau \in \mathrm{Gal}(E/\mathbf{Q})$ for which $\sigma\tau \neq \tau\sigma$. What are the orders of your choice of $\sigma$, $\tau$, and $\sigma\tau$ in the Galois group?

f) Let $K$, $K'$, and $K''$ be the fixed fields of the cyclic groups $\langle\sigma\rangle$, $\langle\tau\rangle$, and $\langle\sigma\tau\rangle$, respectively. Find generators for each of $K$, $K'$, and $K''$.

**Exercise A11.2:** Find all intermediate fields between $\mathbf{Q}$ and $\mathbf{Q}(\mu_{13})$. Describe each such field explicitly as $\mathbf{Q}(\alpha)$ for a suitable $\alpha$. Also show that each $\mathbf{Q}(\alpha)$ is a Galois extension of $\mathbf{Q}$ and determine $G(\mathbf{Q}(\alpha)/\mathbf{Q})$.

**Exercise A11.3:** Consider the 17th cyclotomic field $E = \mathbf{Q}(\zeta)$, where $\zeta = \exp(2\pi i/17)$. Express the extension $E/\mathbf{Q}$ as a tower of quadratic extensions, and find an explicit expression for $2\cos(2\pi/17) = \zeta + \zeta^{-1}$ in terms of square roots (of expressions involving square roots, etc.). Use this to show that the regular 17-gon is constructible with ruler and compass.

**Exercise A11.4:** Given a field $F$, consider the function field $L = F(y_1, y_2, y_3, y_4)$ and the symmetric function field $K = F(s_1, s_2, s_3, s_4)$ where $s_i$ is the $i$th elementary symmetric polynomial in $\{y_1, y_2, y_3, y_4\}$. We know that we can identify $G(L/K)$ with $S_4$. Let $G = D_4$ be the subgroup of $S_4$ consisting of symmetries of the square:

$$G = \{1, (13), (24), (12)(34), (14)(23), (13)(24), (1234), (1432)\}.$$

a) Define $\alpha = y_1y_2 + y_2y_3 + y_3y_4 + y_4y_1$, and $\beta = y_1y_3 + y_2y_4$. Show that $L_G = K(\alpha) = K(\beta)$.

b) Find $\mathrm{irr}(\beta, K)$. (This involves some calculation! Depending on taste, you may wish to use the letters $a, b, c, d$ instead of $y_1, y_2, y_3, y_4$.)

c) Define $H \leq G$ by $H = \{1, (1234), (13)(24), (1432)\}$. (I.e., $H$ is the cyclic subgroup of rotations of the square.) Find some $\gamma \in L$ such that $L_H = K(\gamma)$. There exists a quadratic in $K(\beta)[x]$ which has $\gamma$ as a root; find the other root $\gamma'$ of this polynomial.

d) (A rather serious computational challenge, for extra credit — I suggest you use a symbolic algebra software package.) We know that $\gamma + \gamma'$, $\gamma\gamma'$, and $\alpha$ all belong to $K(\beta)$. Try to express at least one of them in the form $c_r\beta^r + \cdots + c_1\beta + c_0$, with the $c_i \in K$. Alternatively, you can try to express $\beta$ in terms of $\alpha$, using coefficients in $K$.

**Exercise A11.5:** This problem gives you an idea of how a solvable Galois group allows one to solve a polynomial by radicals. We'll sketch the main ideas involved in solving a general cubic equation. We start with a field $F$ in characteristic zero, and we **assume that $F$ contains a**

**primitive 3rd root of unity $\zeta$.** Let $y_1, y_2, y_3$ be three "independent" transcendental elements, and let

$$K = F(s_1, s_2, s_3), \qquad L = F(y_1, y_2, y_3)$$

where $s_1 = y_1 + y_2 + y_3$, $s_2 = y_1y_2 + y_1y_3 + y_2y_3$, and $s_3 = y_1y_2y_3$ are the elementary symmetric polynomials, as usual. Recall that $\Delta = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3) \in L$ is the discriminant.

a) By considering the action of $S_3$ (viewed as the Galois group $G(L/K)$), show that $\Delta^2 \in K$ but $\Delta \notin K$. Deduce that the field $M = K(\Delta)$ is an extension of $K$ by radicals, with $[M : K] = 2$. (I recommend that you avoid trying to explicitly write $\Delta^2$ in terms of $s_1, s_2, s_3$, since the expression is a bit messy; however it is good if you look up the discriminant of a cubic equation for culture.)

b) Now let $\beta = y_1 + \zeta y_2 + \zeta^2 y_3$. Show that $\beta^3 \in M$ but $\beta \notin M$, this time by considering the action of the subgroup $G(L/M)$ of $S_3$. (Do not calculate $\beta^3$ directly, but instead show that every $\sigma \in G(L/M)$ sends $\beta$ to some simple multiple of $\beta$, which allows you to understand the action on $\beta^3$.) Deduce that $L = M(\beta)$ and that $L$ is an extension of $M$ by radicals.

Cultural note: this shows that we can get the roots $y_1, y_2, y_3$ of the polynomial $x^3 - s_1x^2 + s_2x - s_3 \in K[x]$ by taking a square root of an element of $K$ to get $\Delta$, then a cube root of an element of $K[\Delta]$ to get $\beta$, which we can use to express all the elements of $L$, in particular $y_1$, $y_2$, and $y_3$.

**A few more exercises to look at:**

Section 55, exercises 3, 4, 5, 13, 14, 15.

Section 56, exercises 6, 7.

Jacobson, Basic Algebra I, any exercises you like from chapter 4. I particularly encourage you to read sections 4.5–4.11, and 4.16.