Math 242, Topics in Algebra – Spring 2023
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 9, due Tuesday, April 25 at the beginning of class

Exercises from Fraleigh:

Section 51, exercises 1, 4, 11, 14.

Section 53, exercises 1, 2, 3, 4, 5, 7, 8, 10, 16, 20, 21, 23. (Exercises 1–10 are fairly quick).

Additional Exercises (also required):

Exercise A9.1: (Taken from Jacobson, *Basic Algebra I*) Show for yourself (but do not hand in the proof) that the polynomial $f(x) = x^3 + x^2 - 2x - 1$ is irreducible in $\mathbf{Q}[x]$. Let α be a root of f.

a) Show that $\beta = \alpha^2 - 2$ is also a root of f.

b) Show that $\mathbf{Q}(\alpha)$ is a Galois extension of $\mathbf{Q}$.

c) Find the Galois group $G(\mathbf{Q}(\alpha)/\mathbf{Q})$.

Look at, but do not hand in, the following exercises:

Section 51, exercises 2, 3, 9, 10, 12, 13, 15–22.

Section 52, exercises 1, 2, 3, 4, 7, 8.

Section 53, Exercises 6, 9, 11, 12, 13, 24.

"Look At" Exercise L9.1 – not to be handed in: a) Let F be an infinite field, and let V be a finite-dimensional vector space over F. Given finitely many **proper** subspaces $W_1 \subsetneq V$, $W_2 \subsetneq V$, ..., $W_r \subsetneq V$, show that

$$W_1 \cup \cdots \cup W_r \subsetneq V.$$

Hint: Without loss of generality, $V = F^n$. Show that each W_i is contained in a "hyperplane" H_i given by an equation $a_{i1}x_1 + \cdots a_{in}x_n = 0$. Show that some choice of $x_1, \ldots, x_n \in F$ makes all of these equations (for all i) nonzero.

b) Use the above result to deduce the primitive element theorem. (Sketch: let $n = [E : F] = \{E : F\}$, and let $\sigma_1, \ldots, \sigma_n$ be the different F-embeddings of E into $\overline{F}$. For each pair (i, j) with $i \neq j$, show that $W_{(i,j)} = \{\alpha \in E \mid \sigma_i(\alpha) = \sigma_j(\alpha)\}$ is a proper F-subspace of E. Take γ not in the union of the $W_{(i,j)}$.)