Math 242, Topics in Algebra – Spring 2023
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 8, due Tuesday, April 18 at the beginning of class

Exercises from Fraleigh:
Section 50, exercises 2, 4, 6, 7, 8, 9, 16, 21, 22, 25.

Additional Exercises (also required):

Exercise A8.1: Let $K = \mathbf{Q}(\sqrt{5})$, and $E = \mathbf{Q}(\sqrt{2+\sqrt{5}})$. Note that all field extensions here are separable, since we are in characteristic zero.

a) Show that $K \subset E$.

b) Show that K is a splitting field over $\mathbf{Q}$.

c) Show that E is a splitting field over K.

d) Show that E is **not** a splitting field over $\mathbf{Q}$.

e) What is the smallest splitting field over $\mathbf{Q}$ containing E? (This is called the "normal closure" of E.)

Cultural remark: a separable splitting field extension is called a Galois extension. Fraleigh also uses the term "a normal extension", but modern practice is that "normal" just means "splitting field", without requiring separability.

Exercise A8.2: (Part (c) taken from *Algebra* by Michael Artin)

a) Let $E = \mathbf{Q}(\alpha, i)$, where $\alpha = \sqrt[4]{2}$. Prove that E is the splitting field over $\mathbf{Q}$ of the polynomial $f(x) = x^4 - 2$. Compute the degree $[E : \mathbf{Q}]$.

b) Find all the $\mathbf{Q}$-embeddings of E into $\overline{\mathbf{Q}}$, giving their precise effect on α and its conjugates, as well as on i and $\sqrt{2}$. (There are 8 such $\mathbf{Q}$-embeddings.)

c) Factor $f(x) = x^4 - 2$ into irreducible factors over each of $\mathbf{Q}$, $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{2}, i)$, $\mathbf{Q}(\alpha)$, and $\mathbf{Q}(i\alpha)$.

Suggestion: the roots of $f(x)$ are $\{\alpha, i\alpha, -\alpha, -i\alpha\}$. For each field K in the question, determine which of these roots are conjugate to each other over K by using the isomorphism extension theorem.

Look at, but do not hand in, the following exercises:
Section 50, exercises 10, 17, 18, 20, 23, 24.