Math 242, Topics in Algebra – Spring 2023
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 3, due Tuesday, March 7 at the beginning of class

Exercises from Fraleigh:

Section 29, exercises 2, 4, 6, 8, 10, 12, 13, 14, 21, 22, 25, 30, 31. (Exercises 2–14 are pretty quick. For exercises 2, 4, 6, and 8, please also prove that the polynomial you wrote down is irreducible.)

Additional Exercises (also required):

Exercise A3.1: a) Use the Euclidean algorithm (show your work, of course) to find the inverse $(\overline{1234})^{-1}$ of $\overline{1234}$ in the ring $\mathbf{Z}_{4567}$. Try to use as few steps as possible, by taking remainders in $a = bq + r$ satisfying $|r| \leq b/2$. (You can also use the positive remainders, if you wish — it just takes a few more steps, as you can easily verify.)

b) Now use the Euclidean algorithm to find the inverse $(\overline{x^2 + 3x + 1})^{-1}$ of $\overline{x^2 + 3x + 1}$ in the quotient ring $\mathbf{Q}[x]/(x^3 - 3x + 6)$. Your answer should have the form $\overline{ax^2 + bx + c}$ for suitable $a, b, c \in \mathbf{Q}$.

c) Redo part (b) by setting up a system of linear equations for a, b, c and solving the system.

Exercise A3.2: In this exercise, you can accept without proof that $f(x) = x^4 + \overline{2}x^2 + \overline{2}x + \overline{1}$ is irreducible in $\mathbf{Z}_5[x]$. (But do check this for yourself, without handing it in.) Let $F = \mathbf{Z}_5$, and let $E = F(\alpha)$, where α is a root of $f(x)$. Thus $E \cong F[x]/\langle f \rangle$ is a field with $5^4 = 625$ elements.

a) Let $\beta = \alpha^2 + \alpha$. Find $\operatorname{irr}(\beta, F)$, by writing the powers $1, \beta, \beta^2, \beta^3, \beta^4$ in terms of α and finding a linear combination between them with coefficients in F.

b) (challenge) Find a $\gamma \in E$ such that $\operatorname{irr}(\gamma, F)$ has degree 2. (Note that for general E and F with E a 4-dimensional vector space over F, this may not be possible. But it is always possible when E and F are finite fields.)

Look at, but do not hand in, the following exercises:

Section 47, exercises 1, 2, 4, 5, 6, 10, 14, 15.

Section 29, exercises 5, 7, 9, 11, 15, 16, 17, 18, 26, 27, 29, 35, 36, 37.