Math 242, Topics in Algebra – Spring 2023
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 2, due Tuesday, February 28 at the beginning of class

Exercises from Fraleigh:

Section 45, exercises 1–8 (these are quick), 11, 14, 26, 27.
Section 46, exercises 7, 8, 12.

Additional Exercises (also required):

Exercise A2.1: Define integers $a_0, \ldots, a_9$ by the identity

$$a_9x^9 + a_8x^8 + \cdots + a_1x + a_0 = (8x^2 + 10x + 14)(15x^7 + 9x^6 - 12x^4 + 33x + 90).$$

Find $\gcd(a_0, \ldots, a_9)$.

Exercise A2.2: Let R be any commutative ring, and let $a, b \in R$.

a) Recall that $\langle a, b\rangle$ is the subset of R defined by

$$\langle a, b\rangle = \{c \in R \mid \exists s, t \in R \text{ with } c = sa + tb\}.$$

Carefully prove, yet again, that (i) $\langle a, b\rangle$ is an ideal of R, that (ii) both elements a and b belong to $\langle a, b\rangle$, and that (iii) every ideal $I \subset R$ for which $a, b \in I$ satisfies $\langle a, b\rangle \subset I$. (Interpretation: $\langle a, b\rangle$ is the "smallest" ideal containing a and b.)

b) Now assume $q, r \in R$ satisfy $a = bq + r$. Show that $\langle a, b\rangle = \langle b, r\rangle$. (We are not assuming that r is "small" here, but in practice this result will be used when r is the "remainder" when one divides a by b.)

Exercise A2.3: a) Using the Euclidean algorithm, find the GCD of the following two polynomials in $\mathbf{Z}_5[x]$:

$$f = x^6 + 3x^3 + 2, \qquad g = 2x^5 + x^3 + x^2 + 4x.$$

Also express the GCD as a linear combination $sf + tg$ for some choice of $s, t \in \mathbf{Z}_5[x]$.

b) Find the GCD of the above polynomials by first factoring each of f and g into a product of irreducible polynomials.

Look at, but do not hand in, the following exercises:

Section 45, exercises 10, 15, 17, 22, 23, 25, 29, 31, 32.
Section 46, exercises 10, 11, 16, 22, 23, 24.