Math 261 — Fall 2022
Number Theory
`https://sites.aub.edu.lb/kmakdisi/`
Problem set 9, due Friday, November 11 at the beginning of class

Exercise 9.1: Find the continued fraction and all the convergents to the number

$$\frac{489}{292} \approx 1.6746575$$

Note: express the convergents both as fractions and in terms of decimals.

Exercise 9.2: Find the first five convergents to the numbers

$$e \approx 2.718281828, \qquad \pi^4 \approx 97.409091034$$

(As in the previous exercise, express the convergents both as fractions and in terms of decimals.)

Exercise 9.3: Find the repeating continued fraction of $\sqrt{14}$. List the first few convergents A_n/B_n to $\sqrt{14}$ (expressed as fractions) until you reach a convergent for which your calculations show that $\left|A_n/B_n - \sqrt{14}\right| \leq 1/2000$.

(The next exercise asks you to use our theorems to bound the error in a similar situation without doing numerical calculations in $\mathbf{R}$.)

Exercise 9.4: Evaluate the eventually periodic continued fraction

$$\alpha = 1 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\cdots$$

Find the first few convergents until you obtain one for which you can prove, using our theorems from class, that

$$\left|\frac{A}{B} - \alpha\right| \leq \frac{1}{5000}.$$

Exercise 9.5: Let A/B and A'/B' be positive rational numbers (with A, B, A', B' positive integers) satisfying $A/B < A'/B'$ and $A'B - AB' = 1$. Show that if x/y is a positive rational number satisfying $A/B < x/y < A'/B'$, then the denominator y satisfies $y \geq B + B'$. Also show that there does exist a positive rational number of the above form with $y = B + B'$. (Hint: show that $xB - yA > 0$, from which you can then show that $xB - yA \geq 1$; similarly, $yA' - xB' \geq 1$. Then take a suitable linear combination of these two inequalities.)

Conclude the following: given a real number α, let A_{n-1}/B_{n-1} and A_n/B_n be two consecutive convergents in the continued fraction for α, and let $N = B_{n-1} + B_n$. Then one of these two convergents is the best rational approximation to α with denominator less than N. In other words, among all fractions x/y with $y < N$, the difference

$$\left|\frac{x}{y} - \alpha\right|$$

is minimized for one of the choices $x/y = A_{n-1}/B_{n-1}$ or $x/y = A_n/B_n$.

(Note: in fact, the difference is minimized for A_n/B_n; feel free to try to prove this.)