Math 341, Modules and Rings I – Fall 2022
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 1, due Friday, September 9 at the beginning of class

Exercises from Jacobson, BA I:
Section 2.5, exercises 4, 5.
Section 2.15, exercise 5.

Additional Exercises (also required):

Exercise A1.1: Let R be a ring, not necessarily commutative, and let I, J be (two-sided) ideals of R.

a) The "ideal sum" $I + J$ is defined by

$$I + J = \{x \in R \mid \exists i \in I, j \in J \text{ such that } x = i + j\}.$$

The definition is usually abbreviated to $I + J = \{i + j \mid i \in I, j \in J\}$. Show that $I + J$ is an ideal containing I and J.

b) The "ideal product" IJ is defined by

$$IJ = \{x \in R \mid \exists \text{ finitely many } i_1, \ldots, i_n \in I, j_1, \ldots, j_n \in J \text{ such that } x = i_1 j_1 + \cdots + i_n j_n\}.$$

Show that IJ is an ideal contained in $I \cap J$.

c) When R is a PID (if you like, let $R = \mathbf{Z}$), what do the ideal sum and ideal product mean? In other words, if $\langle a\rangle + \langle b\rangle = \langle s\rangle$, and $\langle a\rangle\langle b\rangle = \langle p\rangle$, what are s and p in terms of a and b? What about the intersection $\langle a\rangle \cap \langle b\rangle = \langle i\rangle$?

d) Suppose that R is commutative and that $I + J = R$. Show that $IJ = I \cap J$.

Exercise A1.2: Let F be a field, and let $R = F[x, y]$ be the ring of polynomials in two variables. Show that the ideal $\langle x, y\rangle$ is not principal. Hence R is not a PID.

Cultural remark: $F[x, y]$ is a UFD, as is $F[x_1, x_2, \ldots, x_n]$, or more generally $D[x_1, x_2, \ldots, x_n]$ for D a UFD (for example, $D = \mathbf{Z}$). See Theorem 2.25 in Section 2.16 of BA I.

Exercise A1.3: a) Let R be a PID, and let $I = \langle a\rangle$ be an ideal with a neither zero nor a unit. Consider a finite **strictly** ascending chain of ideals starting at I ("strictly" means consecutive ideals are never equal, and "finite" just means the chain does not go on forever):

$$(*) \qquad I \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n.$$

For a given a, what is the largest possible value of n?

b) Now let $R = \mathbf{Q}[x, y]$. Show that if one starts with the ideal $I = \langle x\rangle$, there exist strictly ascending chains of ideals that can be as long as you like. So the n in $(*)$ is not bounded above in this case. (This also implies that R is not a PID, but the proof in Exercise A1.2 is preferable because it is much more direct.)

Exercise A1.4: Let F be a field. Show that the ring $R = M_n(F)$ of $n \times n$ matrices with entries in F has only two ideals, namely $\{0\}$ and R. If the general case gives you trouble, do at least the case $n = 2$. (Culture: compare this result to BA I, exercises 2.5.4 and 2.5.8.)

Exercise A1.5: Let $a, b, c, d, s, t \in \mathbf{Z}$, and define u, w by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} u \\ w \end{pmatrix}.$$

a) Show that one GCD is a factor of the other: $\gcd(s, t) \mid \gcd(u, w)$.

b) Show that if the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has $\det M = \pm 1$, then in fact $\gcd(s, t) = \gcd(u, w)$.

c) Show more generally that $\gcd(u, w) \mid (\det M) \cdot \gcd(s, t)$.

Remark: this setup generalizes to $\gcd(s_1, \ldots, s_n)$, the GCD of n integers, and $M \in M_n(\mathbf{Z})$.

Look at, but do not hand in:
BA I, 2.6.5, 2.7.10, 2.7.11, 2.15.4, 2.15.11, 2.15.12, 2.15.13–2.15.16 (partial fractions!).