Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 9, due Thursday, November 26 at 2pm via Moodle

Exercises from Jacobson, BA II:

Section 3.2, exercises 2, 3. (For 2: "epimorphism" means "surjective homomorphism", and "monomorphism" means "injective homomorphism".)

Section 3.3, exercises 2, 3 (do these exercises just for modules).

Additional Exercises (also required):

Exercise A9.1: Define a covariant functor $F : \mathbf{Z}\text{-}\mathbf{mod} \to \mathbf{Z}\text{-}\mathbf{mod}$ by the following effect on objects:

$$FM = M[2] := \{x \in M \mid 2x = 0\}.$$

a) How should one define the effect of F on a morphism $f : M \to N$?

b) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. Prove directly that the resulting sequence $0 \to FM' \to FM \to FM''$ is exact.

c) Show that F is naturally isomorphic to a certain Hom functor, and briefly explain why this gives a second way to prove the result in (b).

Exercise A9.2: Let M be an R-module, and let N_1, N_2 be submodules of M. Assume that M/N_1 and M/N_2 are both Noetherian modules. Show that $M/(N_1 \cap N_2)$ is also Noetherian.

Exercise A9.3: Let R be a PID that is not a field.

a) Show that an R-module M has a composition series if and only if M is a finitely generated torsion module.

b) Show that an R-module M is completely reducible (i.e., a direct sum of possibly infinitely many irreducible modules) if and only if M is a torsion module with the property that the annihilator of every nonzero element $x \in M$ is a "squarefree" ideal, i.e., $\operatorname{Ann} x = \langle a \rangle$ with $a = p_1 p_2 \cdots p_r$, a product of prime elements that are not associates of each other. (Note that it is possible for M not to be finitely generated.)

Exercise A9.4: All modules appearing in this exercise have a composition series. Recall that the length $\ell(M)$ of a module is the length of any composition series for M.

a) Show that if $0 \to M' \to M \to M'' \to 0$ is exact, then $\ell(M') - \ell(M) + \ell(M'') = 0$.

b) Show that if $0 \to M_1 \to M_2 \to M_3 \to M_4 \to \cdots \to M_n \to 0$ is exact, then $\ell(M_1) - \ell(M_2) + \ell(M_3) - \ell(M_4) + \cdots \pm \ell(M_n) = 0$.

Exercise A9.5: Let $R = \mathbf{Q}[\lambda]$, let A be a matrix in $M_n(\mathbf{Q})$, and use it to turn $V = \mathbf{Q}^n$ into an R-module as usual. For each of the following choices of A, describe all possible composition series for V, viewed as an R-module.

$$\text{(a) } A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \text{ (b) } A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \text{ (c) } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \text{ (d) } A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise A9.6: (A Noetherian ring) Consider the ring $R = \mathbf{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbf{Z}\}$. (Be sure to check for yourself that R is a subring of $\mathbf{C}$, and therefore a commutative ring, so we do not have to distinguish between left and right ideals. Alternatively, convince yourself that R is isomorphic to $\mathbf{Z}[x]/\langle x^2 + 5\rangle$.)

Note that as a $\mathbf{Z}$-module, R is isomorphic to $\mathbf{Z}^2$. (Also convince yourself of this fact.)

You will need the **norm** of an element $\alpha = a + b\sqrt{-5}$: it is $\operatorname{Nm}(\alpha) = a^2 + 5b^2 \in \mathbf{Z}$. Check for yourself that $\operatorname{Nm}(\alpha\beta) = \operatorname{Nm}(\alpha)\operatorname{Nm}(\beta)$.

a) If $\alpha \in R$ is nonzero, show that the principal ideal $I = \langle \alpha \rangle \subset R$ is also a free $\mathbf{Z}$-module of rank 2. Moreover, show that the quotient ring R/I is finite, with cardinality $|R/I| = \operatorname{Nm}(\alpha)$.

b) Deduce that every nonzero ideal $I \subset R$ is a free $\mathbf{Z}$-module of rank 2, and deduce further that R is a Noetherian ring. Hint for the first part: every nonzero ideal contains a nonzero principal ideal.

c) Let $I = \langle 2, 1 + \sqrt{-5} \rangle$ be the ideal of R generated by 2 and $1 + \sqrt{-5}$. Find a $\mathbf{Z}$-basis for I, and use it to find $|R/I|$. Show that I cannot be a principal ideal.

"Look at" part d) (a nonunique decomposition): Show that R and I are indecomposable as R-modules, that R and I are not isomorphic, and that $R \oplus R \cong I \oplus I$. (One possible isomorphism sends $(a, b) \in R \oplus R$ to $(2a + (1 + \sqrt{-5})b, (1 - \sqrt{-5})a + 2b)$.)

Look at, but do not hand in:

BA II, Section 3.4 and its exercises, especially 1, 3, 4, 7.