Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 8, due Tuesday, November 17 at 2pm via Moodle

Exercises from Jacobson, BA II:

Section 1.1, exercises 3, 4.

Section 1.3, exercise 6.

Section 1.5, exercises 6, 7, 8. In exercise 6, **also show** that the pullback C (if it exists) can be related to the product (if it exists) in the category $\mathbf{C}/B$ from exercise 1.1.3 of the two objects $A_1 \xrightarrow{f_1} B$ and $A_2 \xrightarrow{f_2} B$. The actual product in $\mathbf{C}/B$ is a certain morphism $C \to B$; what is the morphism in question?

Section 3.1, exercises 1, 2, 3, 4.

Additional Exercises (also required):

Exercise A8.1: Given the split exact sequence of R-modules

$$0 \to M_1 \xrightarrow{i_1} M_1 \oplus M_2 \xrightarrow{p_2} M_2 \to 0.$$

Let N be any R-module. Show that the resulting sequences

$$0 \to \mathrm{Hom}_R(N, M_1) \xrightarrow{i_{1*}} \mathrm{Hom}_R(N, M_1 \oplus M_2) \xrightarrow{p_{2*}} \mathrm{Hom}_R(N, M_2) \to 0,$$

$$0 \to \mathrm{Hom}_R(M_2, N) \xrightarrow{p_2^*} \mathrm{Hom}_R(M_1 \oplus M_2, N) \xrightarrow{i_1^*} \mathrm{Hom}_R(M_1, N) \to 0$$

are also (split) exact.

(Culture: this means that for any split exact sequence, applying the $\mathrm{Hom}_R(N, \_\_)$ functor preserves exactness, as does the functor $\mathrm{Hom}_R(\_\_, N)$. Contrast this with the usual "half-exactness" of the Hom functors. Additional "Look At" problem: suppose that an exact sequence $0 \to M' \to M \to M'' \to 0$ has its exactness preserved by either $\mathrm{Hom}_R(M'', \_\_)$ or $\mathrm{Hom}_R(\_\_, M')$. Show then that the original sequence must be split exact.)

Look at, but do not hand in:

BA II, 1.1.5, 1.1.6, 1.2.1, 1.3.1, 1.3.2, 1.3.4, 1.5.1, 1.5.3, 1.5.4, 1.5.5, 1.7.4.

(I particularly recommend that you look at exercises 1.1.5, 1.5.3 and 1.5.4, which describe the product as a terminal object in the category $\mathbf{C}/\{A_1, A_2\}$.)

"Look At" exercise L8.1: Here is an exact sequence of multiplicative groups. Let F be a field, let F^* be its multiplicative group, and consider the following two subgroups of $GL(2, F)$:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in F^*,\, b \in F \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in F \right\}.$$

Then we have an exact sequence (you find the homomorphisms)

$$1 \to N \to G \to F^* \to 1.$$

Show that there exists a group homomorphism $F^* \to G$ that "splits" this exact sequence, but that G is not isomorphic to $N \times F^*$. (It is isomorphic to the **semidirect product** of N and F^* — look it up.)

Further culture: More generally, if $1 \to H_1 \to H \to H_2 \to 1$ is an exact sequence of groups, and if there is a "splitting" homomorphism $H \to H_1$, then in fact $H \cong H_1 \times H_2$. So the situation with split exact sequences of nonabelian groups is slightly different from that with modules.

"Look At" exercise L8.2: An exercise on direct limits (you may also refer to section 2.5 of Jacobson). We first define direct limits: let $\mathbf{C}$ be a category. A *direct system* in $\mathbf{C}$ is a collection of objects X_n in $\mathbf{C}$, indexed by integers $n \geq 1$ (more general index sets are possible), equipped with morphisms $X_m \xrightarrow{f_{m,n}} X_n$, whenever $m \leq n$. The morphisms must satisfy the compatibility conditions

(1) $$f_{m,m} = 1_{X_m} \qquad \text{for all } m,$$

$$(2)\qquad \begin{array}{ccc} X_m & \xrightarrow{f_{m,n}} & X_n \\ & \searrow^{f_{m,k}} & \downarrow f_{n,k} \\ & & X_k \end{array} \quad \text{commutes, whenever } m \le n \le k.$$

Example: in **Ab**, let p be an integer and let $X_n = \mathbf{Z}/p^n\mathbf{Z}$, with $f_{m,n}$ being multiplication by p^{n-m}.

Given an object X of $\mathbf{C}$, together with morphisms $X_n \xrightarrow{f_n} X$ for all n, we say that the f_n are *compatible* with the $f_{m,n}$ if the following diagram commutes whenever $m \le n$:

$$\begin{array}{ccc} X_m & \xrightarrow{f_{m,n}} & X_n \\ & \searrow^{f_m} & \downarrow f_n \\ & & X \end{array}$$

We define the *direct limit* of a directed system to be an object X together with morphisms f_n that are compatible with the $f_{m,n}$, that satisfies the following universal mapping property: given any object Y of $\mathbf{C}$, together with morphisms $X_n \xrightarrow{g_n} Y$ that are compatible with the $f_{m,n}$, then there exists a *unique* morphism $X \xrightarrow{\varphi} Y$, making all the diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & X \\ & \searrow^{g_n} & \downarrow \varphi \\ & & Y \end{array}$$

commute. By the standard argument, if a direct limit exists, it is unique up to canonical isomorphism. Usually the morphisms $f_{m,n}$ are implicit, and one writes $X = \varinjlim X_n$.

Your task in this problem is to show that direct limits exist in $R - \mathbf{mod}$, and to compute $\varinjlim \mathbf{Z}/p^n\mathbf{Z}$ with respect to the morphisms given in the above example.

(Hint: given modules X_n and morphisms $f_{m,n}$, form the disjoint union of the X_n, and consider the equivalence classes of the relation

$$x_m \sim x_n \iff \exists k \ge m, n \text{ with } f_{m,k}(x_m) = f_{n,k}(x_n).$$

Here $x_m \in X_m$ and $x_n \in X_n$. How would you define addition on these equivalence classes, or the product of an element of R by such an equivalence class?)

Remark: If we let the arrows go the other way, we can define the *inverse limit* $\varprojlim X_n$, relative to a compatible family of morphisms $X_m \xleftarrow{f_{m,n}} X_n$. Direct and inverse limits with respect to more general partially ordered sets ("directed sets," as on page 71 of Jacobson) are also common. If p is prime, then the inverse limit $\varprojlim \mathbf{Z}/p^n\mathbf{Z}$ (where the morphisms $\mathbf{Z}/p^m\mathbf{Z} \leftarrow \mathbf{Z}/p^n\mathbf{Z}$ are now the usual reductions modulo p^m) is none other than the ring of p-adic integers $\mathbf{Z}_p$.

"Look At" exercise L8.3: (Exactness of direct limits) In **mod**-R, take for each integer $n \ge 1$ a short exact sequence

$$0 \longrightarrow M'_n \xrightarrow{i_n} M_n \xrightarrow{p_n} M''_n \longrightarrow 0.$$

Assume that whenever $m \le n$, we have homomorphisms $M'_m \xrightarrow{f'_{m,n}} M'_n$, $M_m \xrightarrow{f_{m,n}} M_n$, and $M''_m \xrightarrow{f''_{m,n}} M''_n$, satisfying the compatibility conditions of Exercise A8.1, such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_m & \xrightarrow{i_m} & M_m & \xrightarrow{p_m} & M''_m & \longrightarrow & 0 \\ & & \downarrow f'_{m,n} & & \downarrow f_{m,n} & & \downarrow f''_{m,n} & & \\ 0 & \longrightarrow & M'_n & \xrightarrow{i_n} & M_n & \xrightarrow{p_n} & M''_n & \longrightarrow & 0 \end{array}$$

Then show that

$$0 \longrightarrow \varinjlim M'_n \xrightarrow{i} \varinjlim M_n \xrightarrow{p} \varinjlim M''_n \longrightarrow 0$$

is exact, where i and p are the induced homomorphisms.

Remark: If we were to take inverse limits with respect to suitable families of homomorphisms, we would only get the exact sequence

$$0 \longrightarrow \varprojlim M'_n \longrightarrow \varprojlim M_n \longrightarrow \varprojlim M''_n.$$