Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 7, NOT DUE; "Look At" only before the midterm

Midterm exam: Review the instructions on problem set 6. The written part starts on Monday, November 2 at 8:15am in our usual webex "classroom". The oral part is on Tuesday, November 3, by individual zoom meeting (I will send the coordinates).

The midterm is open book and notes (only Jacobson and your course notes, no other references). However the open book and notes are there only as a backup in case you forget something. You should be able to do the midterm without them.

"Look At" Exercises from Jacobson, BA I:

Look at Section 3.10, exercises 5, 10, 11, 12, 13, 14, 15. (You have already "looked at" a couple of these before.)

Additional "Look At" Exercises:

Exercise L7.1: Let M be a finitely-generated **torsion** module over the PID R.

a) Show that there exists a nonzero $n \in R$ such that $\forall x \in M$, $nx = 0$.

b) Let p be an irreducible element of M, and let $M_p \subset M$ be the p-primary component. We know that $M = M_p \oplus N$ for a suitable complement N of M_p (actually, N is the direct sum of all the q-primary components with q not an associate of p). **Prove** that there exists $h \in R$ with the property that for all $x \in M$, hx is the projection of x to the M_p component.

Hint: the argument is related to the Chinese Remainder Theorem.

Exercise L7.2: Given the following block diagonal matrix $A \in M_5(\mathbf{C})$:

$$A = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ & & & 0 & 1 \\ & & & 0 & 4 \end{pmatrix}.$$

Find the Jordan canonical form of A. (This can be done with almost no calculation. Hint: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is the companion matrix of the polynomial $\lambda^3 - \lambda^2$, and $\begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}$ is the companion matrix of $\lambda^2 - 4\lambda$.)

Exercise L7.3: Let $A \in M_4(\mathbf{C})$ have characteristic polynomial $p_A(\lambda) = (\lambda - 1)^3(\lambda - 2)$.

a) List all possible Jordan canonical forms of A, along with, for each case: (i) the minimal polynomial $m_A(\lambda)$ of A; (ii) the dimension of each eigenspace.

b) In exactly one of your cases above, A is diagonalizable. In that case, what is the corresponding rational canonical form?

Exercise L7.4: (Taken from Hungerford and slightly adapted) Let a, b, c be distinct elements of a field F. Find the characteristic polynomial, minimal polynomial, and rational normal form of the following diagonal matrix:

$$\begin{pmatrix} a & & & & & & \\ & a & & & & & \\ & & a & & & & \\ & & & a & & & \\ & & & & b & & \\ & & & & & b & \\ & & & & & & c \end{pmatrix}$$

Exercise L7.5: Define the homomorphism $\varphi : \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}/20\mathbf{Z}$ by

$$\varphi(a, b, c \bmod 4) = (18a + 24b, a + 10c \bmod 20).$$

Check for yourself, but do not hand in a proof of, the fact that φ is well-defined.

Your task is to find a finite set of generators for each of the following $\mathbf{Z}$-modules, and then to identify the isomorphism type of each $\mathbf{Z}$-module. In other words, exhibit an isomorphism between each module below and an appropriate direct sum of cyclic modules of the form $\mathbf{Z}^r \oplus \mathbf{Z}/\langle d_1 \rangle \oplus \cdots \oplus \mathbf{Z}/\langle d_s \rangle$:

(i) the image $\operatorname{Image} \varphi$,

(ii) the kernel $\operatorname{Ker} \varphi$,

(iii) the cokernel $\operatorname{Coker} \varphi$.