Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 11, NOT DUE; "Look At" only before the final

"Look At" Exercises from Jacobson, BA II:

Section 3.10, exercises 1, 2, 4, 6, 8 (for exercise 4, you can assume that Δ is a field if you like). Also look at exercises 3, 5, 7.

Section 3.11, exercises 1, 2.

Additional "Look At" Exercises:

"Look At" Exercise L11.1: All modules are $\mathbf{Z}$-modules in this exercise.

a) Show that $\mathbf{Q}/\mathbf{Z}$ is divisible, and conclude that it is an injective $\mathbf{Z}$-module.

b) Let M be a module, and let $x_0 \in M$ be **nonzero**. Show that there exists a homomorphism $f : M \to \mathbf{Q}/\mathbf{Z}$ such that $f(x_0) \neq 0$.

"Look At" Exercise L11.2: Let R be a ring in which every ideal is projective (this is a weaker condition than being a PID). Show (by imitating the corresponding proof for PIDs) that every submodule of the free module R^n is isomorphic to a direct sum $I_1 \oplus I_2 \oplus \cdots \oplus I_k$ of ideals $I_1, \ldots, I_k$ with $k \leq n$.

"Look At" Exercise L11.3: Let M be a module over a ring R.

a) Show that there exist free modules F and G, and there exist homomorphisms α and β, so that we have an exact sequence $G \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0$.

b) If R is a Noetherian ring, and M is a finitely-generated R-module, show that we can take F and G to be finitely-generated free modules.

c) Assume we are given another module M' and a homomorphism $\mu : M \to M'$. Assume we also know an exact sequence $G' \xrightarrow{\alpha'} F' \xrightarrow{\beta'} M' \to 0$. (In applications, F' and G' are usually free modules, but we do not need this.) Show that there exist homomorphisms φ, γ such that the following diagram commutes:

$$\begin{array}{ccccccc} G & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & M & \to & 0 \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mu} & & \\ G' & \xrightarrow{\alpha'} & F' & \xrightarrow{\beta'} & M' & \to & 0 \end{array}$$

Hint: Free modules are projective.

Culture, part I: The above is the first step in thinking about (projective or free) "resolutions" of modules. There is a dual notion of "injective resolutions". See Section 6.5 of BA II.

Culture, part II: Look up the concept of "finitely presented module". If in the above F and G are free modules of finite rank for **one** choice of generators and relations, and if F' is of finite rank, then one can show that $\ker \beta'$ is finitely generated, so one can take G' to be a free module of finite rank also.

"Look at" Exercise L11.4: Compute the following tensor products of $\mathbf{Z}$-modules: $\mathbf{Z}/\langle 2\rangle \otimes \mathbf{Z}/\langle 3\rangle$, $\mathbf{Q} \otimes \mathbf{Z}/\langle n\rangle$, $\mathbf{Q} \otimes M$ where M is a finitely-generated $\mathbf{Z}$-module, $\mathbf{Q} \otimes \mathbf{Q}$, $\mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q}$, $\mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q}/\mathbf{Z}$.

"Look at" Exercise L11.5: Let A, B, C be R-modules, and assume given homomorphisms $f : A \to B$ and $g : B \to C$ such that for **every** R-module P, the following sequence is exact:

$$0 \to \mathrm{Hom}_R(C, P) \xrightarrow{g^*} \mathrm{Hom}_R(B, P) \xrightarrow{f^*} \mathrm{Hom}_R(A, P).$$

Show that the sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ must be exact.

Hints: (i) to show g is surjective, take $P = \mathrm{Coker}\, g$, (ii) to show $gf = 0$, take $P = C$, (iii) to show $\mathrm{Ker}\, g \subset \mathrm{Image}\, f$, take $P = \mathrm{Coker}\, f$. Culture: if $0 \to M' \to M \to M'' \to 0$ is exact, apply the above to $A = M' \otimes N$, $B = M \otimes N$, and $C = M'' \otimes N$ to give a different proof that $—\otimes N$ is a right-exact functor.

"Look at" Exercise L11.6: (Faithful flatness) Say R is a commutative ring (although roughly the same proofs work in the noncommutative case). To say that a (left) R-module M is *faithfully flat* means that the sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

is exact *if and only if* the sequence

$$0 \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes N'' \longrightarrow 0$$

is exact. Show that the following are equivalent:

1) M is faithfully flat.

2) M is flat, and $M \otimes N = 0$ implies $N = 0$.

3) M is flat, and $IM \neq M$ for any maximal ideal I of R.

Hints: Recall that IM is the set of finite sums $\sum_{k=1}^{n} i_k m_k$, where $i_k \in I$ and $m_k \in M$; IM is a submodule of M. Also recall that for any ideal J in R, $M \otimes (R/J)$ is isomorphic to M/JM. The isomorphism is canonical, but you don't need that here. You will find it easiest to show $1 \iff 2$ and $2 \iff 3$.)