Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 10, due Thursday, December 3 at 2pm via Moodle

Exercises from Jacobson, BA II:

Section 3.5, exercises 1, 2, 4, 6, 7 (for exercise 4, you may assume that V has finite dimension as an F-vector space).

Section 3.7, exercises 2, 3, 4.

Additional Exercises (also required):

Exercise A10.1: Let R be a ring, and I an ideal of R. Let $M \in R$-**mod**. All tensor products are over R, and you may assume R is commutative if you feel like it.

Your goal in this exercise is to show that $(R/I) \otimes M \cong M/IM$ in two different ways:

a) Using theorems we saw in class (hint: start with the short exact sequence $0 \to I \to R \to R/I \to 0$);

b) By constructing homomorphisms $M/IM \to (R/I) \otimes M$ and $(R/I) \otimes M \to M/IM$, and showing they are inverses.

Exercise A10.2: For a **Z**-module M, define $FM = M/2M$, where, as you might expect, $2M$ is the submodule $2M = \{y \in M \mid \exists x \in M \text{ such that } y = 2x\}$.

a) If $f : M \to N$ is a homomorphism, define a suitable homomorphism $f_* : FM \to FN$ (make sure that f_* is well-defined). Letting $Ff = f_*$, this turns F into a functor from **Z-mod** to itself (but do not check the properties of a functor; we have other things to do in this exercise).

b) Give a direct proof that if $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact, then

$$M'/2M' \xrightarrow{f_*} M/2M \xrightarrow{g_*} M''/2M'' \to 0$$

is exact. (Note that f_* might not be injective.)

c) Show that the functor F is **naturally** isomorphic to the functor $G = \mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} -$ (the notation for G means as usual that $GM = \mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} M$).

(Culture: this, plus what we know about tensor products, gives a different proof of part (b).)

Exercise A10.3: Let R, S be rings, and let $\phi : R \to S$ be a ring homomorphism. View $S = S_R$ as a right R-module via $s \cdot r := s\phi(r)$.

a) Suppose that $M = {}_RM$ is a left R-module, and define $M' = S \otimes_R M$. Let $s' \in S$. **Show** that there exists a homomorphism of abelian groups $T_{s'} : M' \to M'$ with the property that on generators $s \otimes x$ (with $s \in S$ and $x \in M$), we have $T_{s'}(s \otimes x) = (s's) \otimes x$.

b) Show that above turns $M' = S \otimes_R M$ into a left S-module, via $s'(y) = T_{s'}(y)$ for all $s' \in S$ and $y \in M'$.

c) Show that $S \otimes_R R \cong S$ **as S-modules**.

Look at, but do not hand in:

BA II, Exercises 3.8.3, 3.8.4 (refer to Section 1.8 for the definition of an adjoint functor), 3.9.11.

"Look At" exercise L10.1: Show that tensor products are compatible with direct limits. More precisely, consider a direct limit $\varinjlim M_n$ made from right R-modules M_n and compatible maps $f_{m,n}$. Also consider a left R-module N. Show that

$$(\varinjlim M_n) \otimes_R N \cong \varinjlim (M_n \otimes_R N).$$

Hint: construct the maps in both directions, and show they are inverses to each other. For the forward map, first construct a balanced map $(\varinjlim M_n) \times N \to \varinjlim (M_n \otimes_R N)$.

Note: no penalty for doing this exercise entirely in the case where R is commutative, if it makes things easier for you.

"Look At" exercise L10.2: (The Ext functor) Fix a module M in **mod**-R, and define, for a module N, $\mathrm{Ext}_R(M, N)$ to be the set of extensions of M by N. This means that elements of $\mathrm{Ext}_R(M, N)$ are equivalence classes of short exact sequences $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$, where we define the sequence $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{p} M \longrightarrow 0$ to be equivalent to $0 \longrightarrow N \xrightarrow{i'} E' \xrightarrow{p'} M \longrightarrow 0$ if there exists $E \xrightarrow{f} E'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & E & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow 1_N & & \downarrow f & & \downarrow 1_M & & \\ 0 & \longrightarrow & N & \xrightarrow{i'} & E' & \xrightarrow{p'} & M & \longrightarrow & 0 \end{array}$$

commute. (f is necessarily an isomorphism, as you showed in BA II, Exercise 3.1.4.) Show that $\mathrm{Ext}_R(M, -)$ is a covariant functor of N. (Hint: pushout.)

Take a surjective homomorphism $F \xrightarrow{\pi} M$, where F is a free R-module, i.e. a direct sum of (possibly infinitely many) copies of R. (This can always be arranged, by picking some set of generators for M, and taking one copy of R for each generator.) Let K be the kernel of π, so that $0 \longrightarrow K \xrightarrow{\iota} F \xrightarrow{\pi} M \longrightarrow 0$ is exact. Then show that $\mathrm{Ext}_R(M, N)$ is naturally isomorphic (as a set) to the quotient group $\mathrm{Hom}(K, N)/\iota^*(\mathrm{Hom}(F, N))$. Here ι^* is the map from $\mathrm{Hom}(F, N)$ to $\mathrm{Hom}(K, N)$.

Note: I'm only asking you to consider $\mathrm{Ext}_R(M, N)$ as a functor in N. As you may have guessed, it's also a contravariant functor in M. The isomorphism between $\mathrm{Ext}_R(M, N)$ and $\mathrm{Hom}(K, N)/\iota^*(\mathrm{Hom}(F, N))$ defines a group structure on $\mathrm{Ext}_R(M, N)$, which can be defined without reference to F and π; see exercise 6.7.4 of BA II.