Math 219, Linear Algebra I — Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 8, due Thursday, November 12 at 2pm via Moodle

Quiz 2: The written part of Quiz 2 will take place during class time on **Monday, November 16**. The oral component will be on Tuesday, November 17.

Exercises from Corwin-Szczarba:

Section 4.7, exercises 15, 16.

Section 7.1, exercises 2, 5, 6 (for exercise 2, write n for $\dim V$). Note: "The matrix of $T : V \to V$ relative to the basis α" means ${}_\alpha[T]_\alpha$.

Section 7.2, exercises 1, 5, 9, 10.

Section 7.3, exercises 1, 2.

Additional Exercises (also required):

Exercise A8.1: In $\mathbf{R}^3$, let $L = \operatorname{span}\{(1,1,0)\}$, and let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a rotation by $\pi/4$ about the axis L. (There are two choices of T, depending on the direction of rotation; pick one that suits you.) You may accept the fact that T is a linear transformation. Find the matrix of T. (Suggestion: find an orthonormal basis $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$ of $L^\perp$, and decompose vectors into their L and $L^\perp$ components. Recall that you know how to rotate vectors in the plane spanned by $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$.)

Exercise A8.2: We define a linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_2$ by $T(f) = f(x) + f(1-x)$. For example, $T(x^3) = x^3 + (1-x)^3 = 1 - 3x + 3x^2$.

a) Write the matrix $M = {}_\beta[T]_\alpha$ of T with respect to the bases $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2\}$.

b) Find bases for each of $\ker T$ and $\operatorname{Image} T$.

c) Find new bases α' for $\mathcal{P}_3$ and β' for $\mathcal{P}_2$ such that ${}_{\beta'}[T]_{\alpha'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M'$.

d) Find explicit invertible matrices P and Q such that $M = PM'Q$.

Exercise A8.3: Define $T : \mathcal{P}_3 \to \mathbf{R}^2$ by $T(f(x)) = \begin{pmatrix} f(2) \\ f(3) \end{pmatrix}$. For example, $T(x^2+1) = \begin{pmatrix} 2^2+1 \\ 3^2+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$. Thus T is a linear transformation (check this).

a) Find the matrix ${}_{\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}}[T]_{\{1,x,x^2,x^3\}}$, where $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}$ is as usual the standard basis of $\mathbf{R}^2$.

b) Find a basis for $\ker T$.

c) Find a new basis α for $\mathcal{P}_3$ and a new basis β for $\mathbf{R}^2$ such that ${}_\beta[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

d) Find explicit invertible matrices P and Q as in the previous exercise.

Look at, but do not hand in:

Section 4.7, exercises 19, 20, 21.

Section 7.1, exercises 3, 4, 13, 14.

Section 7.2, exercises 2, 3, 4, 8, 11, 12.

"Look At" exercise L8.1: Find an orthonormal basis for each of the kernel and image of the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^4$ given by the matrix

$$A_T = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & 2 & 3 \end{pmatrix}.$$

Suggestion: first find any basis, then use Gram-Schmidt. Be prepared for some calculation.

"Look At" exercise L8.2: Let V be an inner product space, and let $\{\vec{\mathbf{u}}_1, \dots \vec{\mathbf{u}}_n\}$ be an orthonormal basis of V. Show that for all $\vec{\mathbf{w}} \in V$, we have

$$\|\vec{\mathbf{w}}\|^2 = \langle \vec{\mathbf{w}}, \vec{\mathbf{u}}_1 \rangle^2 + \langle \vec{\mathbf{w}}, \vec{\mathbf{u}}_2 \rangle^2 + \cdots + \langle \vec{\mathbf{w}}, \vec{\mathbf{u}}_n \rangle^2.$$

"Look At" exercise L8.3: Let V be a finite-dimensional inner product space, and let W be a subspace of V. Define a linear transformation $T : V \to V$ by $T(\vec{\mathbf{v}}) = \vec{\mathbf{v}} - \frac{1}{2}\operatorname{Proj}_W(\vec{\mathbf{v}})$.

a) Show that for all $\vec{\mathbf{v}}$, we have $\frac{1}{2}\|\vec{\mathbf{v}}\| \le \|T(\vec{\mathbf{v}})\| \le \|\vec{\mathbf{v}}\|$.

b) Show that T is a bijection.