Math 219, Linear Algebra I — Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 6, due Friday, October 30 at 2pm via Moodle

Exercises from Corwin-Szczarba:
Section 4.3 exercises 1b, 2e, 3cd, 4.
Section 4.4, exercises 14, 15. (These use the linear extension theorem.)
Section 4.5, exercises 1abc, 7, 8.

Additional Exercises (also required):

Exercise A6.1: Let V be finite-dimensional, and let $W \subset V$ with $W \neq V$ be a proper subspace of V. Fix a vector $\vec{\mathbf{v}}_0 \in V$ such that $\vec{\mathbf{v}}_0 \notin W$. Show that there exists a linear transformation $T : V \to \mathbf{R}$ for which $T(\vec{\mathbf{v}}_0) = 1$ and $W \subset \ker T$.

(Cultural note: this shows that W is the intersection of all hyperplanes containing it.)

Hint: choose a basis for V in a suitable way and use the linear extension theorem.

Exercise A6.2: Let V and W be finite-dimensional vector spaces. Let $\vec{\mathbf{v}} \in V$ and $\vec{\mathbf{w}} \in W$ be **nonzero** vectors. Show that there exists a linear transformation $T : V \to W$ such that $T(\vec{\mathbf{v}}) = \vec{\mathbf{w}}$ and $\dim \operatorname{Image} T = 1$.

Exercise A6.3: Use inner products to prove that the three altitudes of a triangle ABC intersect in a point. (An *altitude* of ABC is a line through one of the points that is perpendicular to the line through the other two; so H lies on the altitude from A iff $\vec{AH} \perp \vec{BC}$.)

Hint: recall that for points P, Q we have $\vec{PQ} = \vec{OQ} - \vec{OP}$, where O is the origin of the plane. Be sure you understand this identity geometrically. It follows algebraically from $\vec{OP} + \vec{PQ} = \vec{OQ}$.

Look at, but do not hand in:
Section 4.3, exercises 5, 6, 7.

"Look At" exercise L6.1, not to be handed in: This exercise sketches a somewhat artificial proof that the row rank and the column rank of a matrix are equal. Recall that you may not use this result until we prove it "officially" later in the course. In this exercise, A is a $p \times n$ matrix with rows $R_1, \ldots, R_p$ and columns $C_1, \ldots, C_n$. Here is an example with a 3×5 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 102 & 202 & 302 & 402 & 502 \end{pmatrix},$$

$$R_1 = (1, 2, 3, 4, 5), \qquad R_2 = (1, 1, 1, 1, 1), \qquad R_3 = (102, 202, 302, 402, 502),$$

$$C_1 = \begin{pmatrix} 1 \\ 1 \\ 102 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 2 \\ 1 \\ 202 \end{pmatrix}, \qquad \ldots, \qquad C_5 = \begin{pmatrix} 5 \\ 1 \\ 502 \end{pmatrix}.$$

a) Viewing $A = A_T$ for a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^p$, assume that some row R_i is a linear combination of the other rows. (For example, in the matrix above, take $i = 3$ and $R_3 = 100R_1 + 2R_2$.) Let B be the $(p-1) \times n$ matrix obtained by removing row R_i from A. (In our example, $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.) Viewing B as the matrix of a linear transformation $U : R^n \to R^{p-1}$, show that $\ker T = \ker U$.

Hint: why are the systems of linear equations for $\ker T$ and $\ker U$ equivalent?

From now on, we will just write $\ker A$ and $\ker B$ for matrices, instead of mentioning the linear transformations. So $\ker A$ is what we previously called $\ker T$.

b) Suppose the row rank of A is r. Show that $\dim \ker A \leq n - r$.

Hint: remove "redundant" rows from A "without changing its kernel", until you obtain an $r \times n$ matrix M with the same kernel. Show using Rank-Nullity that the kernel of (the linear transformation given by) M has dimension at least $n - r$.

(Remark: $r \leq n$, because the rows belong to $\mathbf{R}^n$. Also note that once we finish the exercise, we will see that $\dim \ker A = n - r$.)

c) Deduce from part (b) and Rank-Nullity for (the transformation given by) A that $r \geq$ the column rank of A. Thus "row rank $\geq$ column rank".

d) By applying part (c) to the row and column ranks of the transposed matrix A^t, show that the row rank of the original matrix A is $\leq$ the column rank of the original matrix A. Thus, from (c) and (d), the row and column ranks of A are equal.