Math 219, Linear Algebra I — Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 10, due Saturday, December 5 at 2pm via Moodle

Exercises from Corwin-Szczarba:
Section 10.3, exercises 2, 3, 6, 7.

Additional Exercises (also required):

Exercise A10.1: a) Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix}$.
b) Show that A is diagonalizable.
c) Find $\operatorname{trace}(A^7)$.

Exercise A10.2: Diagonalize the following three matrices over $\mathbf{C}$:

$$\begin{pmatrix} 0 & 3-9i & 1 \\ 1 & 2+3i & 0 \\ 0 & 0 & 1+2i \end{pmatrix}, \quad \begin{pmatrix} 0 & -10 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & -2i \\ 1 & -1 & -2i \\ 2i & 2i & 2 \end{pmatrix}.$$

Exercise A10.3: Let V be a finite-dimensional inner product space, and let $P : V \to V$ be a **self-adjoint** linear transformation such that $P^2 = P$.
a) Why is P diagonalizable?
b) Show that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.
c) Show that P is the orthogonal projection onto a certain subspace W of V.

Exercise A10.4: Find a basis γ with respect to which **both** of the following linear transformations on $\mathbf{R}^3$ become (simultaneously) diagonalized (the matrices below are the matrices with respect to the standard basis):

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Cultural note: A necessary condition for two linear transformations to be simultaneously diagonalized is for them to commute, i.e., $S \circ T = T \circ S$. But this is not sufficient. (Challenge: prove these statements.)

Exercise A10.5: Show that the complex matrix $A = \begin{pmatrix} 2 & i \\ i & 0 \end{pmatrix}$ is **not** diagonalizable, even though $A^{tr} = A$. Why does this not contradict the statement of the spectral theorem?

Exercise A10.6: Let $\mathcal{P}_3$ be as usual the space of polynomials $f(x)$ of degree at most 3. Define a linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ by $T(f) = (x+1)^2 f'' - 4xf' + 6f$.
a) Find the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.
(To check your results: you should get $T(1) = 6$, $T(x) = 2x$, $T(x^2) = 4x + 2$, and $T(x^3) = 12x^2 + 6x$.)
b) Find a basis for each of $\operatorname{Image} T$ and $\ker T$. Justify your reasoning. Make sure you give elements of $\mathcal{P}_3$.
c) Find the eigenvalues of T, and, for each eigenvalue, find one eigenvector. Again, these should be elements of $\mathcal{P}_3$.
d) Show that T is not diagonalizable.

Look at, but do not hand in:
Section 10.1, exercises 1, 2, 3, 4, 5, 6, 7, 8.
Section 10.2, exercises 3, 5 (note correction), 6 (note correction), 7, 8, 10, 19 (note correction), 18 (in that order), 21.
Correction for exercises 10.2.5 and 10.2.6: the matrices in question must be **square**, so both problems should say "$A \in M(n, n, \mathbf{C})$" instead of "$A \in M(n, m, \mathbf{C})$".
Correction for exercise 10.2.19: the first term inside the parentheses should be $\|\vec{\mathbf{v}} + \vec{\mathbf{w}}\|^2$, not $\|\vec{\mathbf{v}} + \vec{\mathbf{v}}\|^2$.
Section 10.4, exercise 1 (feel free to try a couple of others from this section).
Section 10.6, exercises 1cg, 2, 4, 5, 8, 9, 10, 11, 12, 13.
Section 7.5, exercises 9, 10, 11.
Section 10.7, exercises 1, 2. (Read in particular Theorem 7.2 and the subsequent discussion until just before Proposition 7.3 for Cramer's rule and its application to A^{-1}.)

"Look At" Exercise L10.1: Which of the following matrices are equivalent? Which are similar? Explain.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

"Look At" Exercise L10.2: Let V be a finite-dimensional inner product space, and let $T : V \to V$ be a linear transformation.

a) Use the spectral theorem to show that if T is self-adjoint, then $id_V + T^2 : V \to V$ is an invertible linear transformation.

b) Give an example of V and T such that $id_V + T^2$ is not invertible. (Of course, such a T cannot be self-adjoint.) Preferably find such an example over $\mathbf{R}$; if you have trouble finding the example, then just settle for an example over $\mathbf{C}$, which is easier.

"Look At" Exercise L10.3: Let V be a finite-dimensional **complex** inner product space, and let $T : V \to V$ be a unitary transformation. Recall that this means that $T^* = T^{-1}$, which is a fancy way of saying that T is an isometry: for all $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$, $\langle T(\vec{\mathbf{v}}), T(\vec{\mathbf{w}})\rangle = \langle \vec{\mathbf{v}}, \vec{\mathbf{w}}\rangle$.

a) Show that if λ is a (complex) eigenvalue of T, then $|\lambda| = 1$.

b) Show that T is diagonalizable, by imitating the proof of the spectral theorem.

"Look At" Exercise L10.4: Consider the complex matrix $M = \begin{pmatrix} 3 & 3+i \\ 3-i & 6 \end{pmatrix}$.

a) Why do we know without any calculation that M is diagonalizable?

b) Find an **orthonormal** basis $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$ of $\mathbf{C}^2$ consisting of eigenvectors of M, and find the corresponding eigenvalues. (Note: do this by the "usual" way.)

c) For $z_1, z_2 \in \mathbf{C}$, let

$$f(z_1, z_2) = 3z_1\overline{z_1} + (3+i)z_2\overline{z_1} + (3-i)z_1\overline{z_2} + 6z_2\overline{z_2}.$$

Find explicit constants $C_1, C_2 > 0$ for which you can show that for all $z_1, z_2 \in \mathbf{C}$, we have $C_1(|z_1|^2 + |z_2|^2) \le f(z_1, z_2) \le C_2(|z_1|^2 + |z_2|^2)$. Suggestion: write $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = w_1\vec{\mathbf{u}}_1 + w_2\vec{\mathbf{u}}_2$.

Note: if the use of complex numbers throws you off in this exercise, do it first for the real matrix $M = \begin{pmatrix} 12 & 3 \\ 3 & 4 \end{pmatrix}$, and the function $f(x_1, x_2) = 12x_1^2 + 3x_2x_1 + 3x_1x_2 + 4x_2^2 = 12x_1^2 + 6x_1x_2 + 4x_2^2$, with $x_1, x_2 \in \mathbf{R}$. Further hint: $f(\vec{\mathbf{v}}) = \langle M\vec{\mathbf{v}}, \vec{\mathbf{v}}\rangle$.

"Look At" Exercise L10.5: Let V be a finite-dimensional inner product space, and assume given a **self-adjoint** linear transformation $T : V \to V$ such that $T^3 = T$.

a) Show that the only possible eigenvalues of T are $\lambda = 0$, 1, or -1. As usual, let us call the corresponding eigenspaces V_0, V_1, V_{-1}. (It is possible that some of these eigenspaces are just $\{\vec{\mathbf{0}}\}$: for example, if $T = id_V$, then $V_0 = V_{-1} = \{\vec{\mathbf{0}}\}$.)

b) Show that every vector $v \in V$ has a decomposition $v = v_0 + v_1 + v_{-1}$, with $v_0 \in V_0$, $v_1 \in V_1$, and $v_{-1} \in V_{-1}$.

c) Define a linear transformation $P : V \to V$ by $P = (1/2)(T^2 + T)$. Show that P is the orthogonal projection onto V_1.

d) Find another "polynomial" $Q = aT^2 + bT + cI$ for suitable $a, b, c \in \mathbf{R}$, such that Q is the orthogonal projection onto V_0.

"Look At" Exercise L10.6: A proof of the spectral theorem for commuting self-adjoint linear transformations, and similarly for commuting normal linear transformations. Throughout this exercise assume that V is a finite-dimensional complex vector space.

a) If $T, U : V \to V$ are commuting linear transformations (so $T \circ U = U \circ T$), show that T and U have a common eigenvector. (Hint: let λ be an eigenvalue of T [remember, the field of scalars is $\mathbf{C}$] and show that the nonzero eigenspace $V_\lambda = \ker(\lambda I - T)$ is invariant (i.e., stable) under U, and hence [why?] contains an eigenvector for U.)

b) If, furthermore, T and U are self-adjoint, show that V has an orthonormal basis of simultaneous eigenvectors for T and U. Generalize to an arbitrary number of commuting self-adjoint linear transformations. Bonus: show that the result still holds over $\mathbf{R}$, and, whether over $\mathbf{R}$ or over $\mathbf{C}$, that it still holds even if one has an infinite set of commuting self-adjoint linear transformations.

c) Back to the case of $\mathbf{C}$, Assume that T is normal; this means that T and T^* commute. In this case, show that there exists an orthonormal basis of eigenvectors for T. (Hint: show that $T + T^*$ and $i(T - T^*)$ are commuting self-adjoint linear transformations.) Bonus: generalize to an arbitrary number of commuting normal linear transformations.