Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 5, due Tuesday, October 20 at 2pm via Moodle

Exercises from Jacobson, BA I:
Section 3.7, exercises 1, 2, 4, 5, 6, 8.
Section 3.8, exercise 1.

Additional Exercises (also required):

Exercise A5.1: Let V be a five-dimensional vector space over $\mathbf{Q}$ with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_5\}$. We are given a linear transformation $T : V \to V$ such that

$$\begin{aligned} T(\mathbf{x}_1) &= \mathbf{x}_4, \\ T(\mathbf{x}_2) &= 0, \\ T(\mathbf{x}_3) &= \mathbf{x}_2 - \mathbf{x}_3 - 2\mathbf{x}_4 - \mathbf{x}_5, \\ T(\mathbf{x}_4) &= \mathbf{x}_3 + \mathbf{x}_5, \\ T(\mathbf{x}_5) &= \mathbf{x}_3 - 2\mathbf{x}_4 + \mathbf{x}_5. \end{aligned}$$

a) Write the matrix of T with respect to the given basis, and use it to show that the characteristic polynomial $p_T(\lambda)$ is equal to $\lambda^5 + 4\lambda^3$. Viewing V as a module over $\mathbf{Q}[\lambda]$, verify that for all $\mathbf{v} \in V$, $(\lambda^5 + 4\lambda^3)\mathbf{v} = 0$. This is an example of the Cayley-Hamilton theorem.

b) Find a $\mathbf{Q}$-basis for the subspaces $V_1 = \{\mathbf{v} \in V \mid \lambda^3\mathbf{v} = 0\}$ and $V_2 = \{\mathbf{v} \in V \mid (\lambda^2 + 4)\mathbf{v} = 0\}$. Show directly that $V = V_1 \oplus V_2$.

(Note: this is a general phenomenon related to the Chinese Remainder Theorem, because $\gcd(\lambda^3, \lambda^2 + 4) = 1$.)

c) By trial and error, write each of V_1 and V_2 as a direct sum of cyclic modules. (Trial and error in this case is easier than the general method we will discuss, involving a matrix A with entries in $\mathbf{Q}[\lambda]$.)

Exercise A5.2: Show that the following two matrices are equivalent over $\mathbf{Z}$, and find explicit matrices $P, Q \in GL_2(\mathbf{Z})$ with $B = PAQ$:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 21 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Culture: think about the relation of this with the Chinese remainder theorem.)

Exercise A5.3: a) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \in M_{2\times 4}(\mathbf{Z})$, find explicit **elementary** matrices $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_s$ for which

$$P_1 \cdots P_r A Q_1 \cdots Q_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}.$$

(Note: do not compute the products $P_1 \cdots P_r$ or $Q_1 \cdots Q_s$; just give the individual P_i and Q_j.)

b) What is the structure of the quotient module $\mathbf{Z}^4 / \langle (1,2,3,4), (5,6,7,8) \rangle$?

c) Explain why $\langle (1,2,3,4), (5,6,7,8) \rangle$ is a free $\mathbf{Z}$ module, and yet one cannot extend the "linearly independent set" $\{(1,2,3,4), (5,6,7,8)\}$ to a basis of the free module $\mathbf{Z}^4$.

Look at, but do not hand in:
BA I, 3.8.3, 3.8.4, 3.8.5, 3.8.6 (the set 3.8.4–3.8.6 is very highly recommended).