Math 341, Modules and Rings I – Fall 2020
Course website: https://sites.aub.edu.lb/kmakdisi/
Problem set 4, due Tuesday, October 13 at 2pm via Moodle

Exercises from Jacobson, BA I:
Section 3.5, exercises 2, 3, 5.
Section 3.6, exercises 1, 2, 3.

Additional Exercises (also required):

Exercise A4.1: Let M be an R-module, and let $N \subset M$ be a module such that both N and the quotient M/N are finitely generated. Show that M is finitely generated.

Exercise A4.2: Let R be an arbitrary ring, and let M be an R-module. We are given that M contains **exactly one** nontrivial proper submodule N, meaning $0 \subsetneq N \subsetneq M$. (Example: $R = \mathbf{Z}$, $M = \mathbf{Z}/\langle 4 \rangle$, and $N = \{\bar{0}, \bar{2}\}$.)
a) Show that N and M/N are both irreducible modules.
b) Show that M is a cyclic R-module.
c) If R is a PID, what is the structure of M?

Exercise A4.3: We take $R = F$ to be a field, so modules are the same thing as vector spaces. Let V be a finite-dimensional vector space over the field F, and let $W \subset V$ be a subspace. (Note: parts (i) and (ii) generalize to infinite-dimensional vector spaces, but you would need Zorn's Lemma to do the proof in the general case. Feel free to do so if you are so inclined.)
(i) Show that V and W are free F-modules.
(ii) Show that there exists a subspace $W' \subset V$ such that $V = W \oplus W'$ (internal direct sum) — in other words, such that W' is a complementary subspace to W.
(iii) In the special case $F = \mathbf{R}$ and $V = \mathbf{R}^5$, let

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} \in \mathbf{R}^5 \;\middle|\; \begin{array}{r} x + y + z + t \qquad = 0 \\ x + y + \qquad 2t + w = 0 \\ 3x + 5y + 4z + 6t + w = 0 \\ -2x + 2y - 2z + 6t + 4w = 0 \end{array} \right\}.$$

Find a basis for a subset W' as in (ii).

Exercise A4.4: A question about the $\mathbf{Z}$-modules $M = \mathbf{Z}^2/\langle (2,0), (0,10) \rangle$ and $N = \mathbf{Z}^2/\langle (2,10) \rangle$.
a) Show that M is finite, and find (with justification) two elements $v, v' \in M$ such that M is the internal direct sum $M = \langle v \rangle \dot{+} \langle v' \rangle$.
b) Show that N is infinite, and once again find (with justification) two elements $w, w' \in N$ with $N = \langle w \rangle \dot{+} \langle w' \rangle$.

Exercise A4.5: Let F be a field, and let R be the subring of $M_3(F)$ consisting of all matrices of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix}, \qquad a, b, c, d, e, f, g \in F.$$

View F^3 (column vectors) as a left R-module in the obvious way.
Show that F^3 has exactly one nontrivial R-submodule M, and show that M does **not** have a complementary R-submodule. (Caution: if we only view M as an F-subspace, there do exist complementary F-subspaces.)

Look at, but do not hand in:
BA I, 3.5.1, 3.5.4.

Additional "look at" exercise L4.1, not to be handed in: The following are submodules of $\mathbf{Z}^2$ or $\mathbf{Z}^4$; hence they are free also. Find a basis for each submodule.

$$\begin{aligned} M_1 &= \langle (10,20), (40,50), (70,80), (0,15) \rangle, \\ M_2 &= \{(x,y,z,w) \mid x + y + z + w = x + 3y + 5z + 7w = 0\} \\ M_3 &= \{(x,y) \mid x + 2y \equiv 2x + y \equiv 0 \pmod 9\}, \\ M_4 &= \{(x,y,z,w) \mid x + y + z + w = 6x + 10y + 15z + 30w = 0\}. \end{aligned}$$