Math 219, Linear Algebra I — Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 5, due Thursday, October 22 at 2pm via Moodle

Exercises from Corwin-Szczarba:

Section 4.2, exercise 6 (make use of Rank-Nullity, here and below, to minimize the work involved).

Section 4.4, exercises 2, 7, 11bcd (find a basis for the kernel and the image each time), 13, 16.

Hint for exercise 7: start with a basis $\{\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k\}$ for $V_1 \cap V_2$, and then extend it once to a basis $\{\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k, \vec{\mathbf{y}}_1, \dots, \vec{\mathbf{y}}_\ell\}$ for V_1 and again to a basis $\{\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k, \vec{\mathbf{z}}_1, \dots, \vec{\mathbf{z}}_m\}$ for V_2. What do you think will be a basis for $V_1 + V_2$? Prove it and deduce the formula about dimensions.

Additional Exercises (also required):

Exercise A5.1: a) Show that $\{\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}\}$ is a basis for $\mathbf{R}^3$.

b) Deduce that T is surjective, where $T : \mathbf{R}^5 \to \mathbf{R}^3$ is the linear transformation given by the matrix

$$A_T = \begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ 1 & 0 & 0 & 1 & 5 \\ 9 & 1 & 2 & 4 & 9 \end{pmatrix}.$$

c) Find the dimension of $\ker T$ **without** doing any detailed calculations.

Exercise A5.2: Consider the linear transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by $T(f) = f + f'$. Show that T is bijective without necessarily finding the inverse T^{-1}.

Exercise A5.3: Define linear transformations $T, S : \mathcal{M}_{2,2} \to \mathcal{M}_{2,2}$ by

$$T(A) = A \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \qquad S(A) = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} A.$$

a) Verify that T and S are indeed linear transformations.

b) Find a basis for each of $\ker T$, $\operatorname{Image} T$, $\ker S$, and $\operatorname{Image} S$, and compare to the statement of Rank-Nullity.

Look at, but do not hand in:

Section 4.4, exercises 1, 3, 4, 5, 6, 8, 10 (note for exercise 10: this refers to exercise 17 of Section 2.4, not to exercise 18, which does not exist).

"Look At" Exercise L5.1, not to be handed in: This exercise gives a different "geometric" proof of the key **Lemma:** Let $V = \operatorname{span}\{\vec{\mathbf{w}}_1, \dots, \vec{\mathbf{w}}_n\}$. If $S = \{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ with $k > n$, then S is linearly dependent. (Our proof in class was "algebraic", and used systems of linear equations.)

Sketch: We prove this by induction on n. Treat the base case yourself. Argue the inductive step as follows: write $\vec{\mathbf{v}}_j = b_{j1}\vec{\mathbf{w}}_1 + \cdots + b_{jn}\vec{\mathbf{w}}_n$, as in the proof in class. Define $V^* = \operatorname{span}\{\vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3, \dots, \vec{\mathbf{w}}_n\}$, so V^* can be generated by $n^* = n - 1$ vectors, and you know the statement already for V^* (i.e., if $k^* > n^*$, then any set of k^* vectors in V^* is linearly dependent). Argue separately in the two cases where $b_{j1} = 0$ for all j and in the case where one of the b_{j1}'s is nonzero. For example, if $b_{11} \neq 0$, make vectors $\vec{\mathbf{v}}_j^* = \vec{\mathbf{v}}_j - (b_{j1}/b_{11})\vec{\mathbf{v}}_1$ for $2 \leq j \leq k$, and show that the vectors $\vec{\mathbf{v}}_2^*, \dots, \vec{\mathbf{v}}_k^*$ are linearly dependent. Then use this to show that the original vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ are dependent. For geometric intuition, you should visualize $\vec{\mathbf{v}}_j^*$ as being the component of $\vec{\mathbf{v}}_j$ in V^*, after we have removed the component that was parallel to $\vec{\mathbf{v}}_1$.

"Look At" Exercise L5.2, not to be handed in: Let $T : V \to V$ be a linear transformation. We say that T is **nilpotent** if for some $k \geq 1$, T^k is the zero linear transformation. (Here $T^k = T \circ T \circ \cdots \circ T$, for a total of k times.)

a) Show that if T is nilpotent, and $V \neq \{\vec{\mathbf{0}}\}$, then T cannot be injective.

b) Show that if $T : \mathbf{R}^3 \to \mathbf{R}^3$ is nilpotent, then in fact $T^3 = \mathbf{0}$ (no higher power of T is needed). Generalize to the case of $T : V \to V$ where V is finite-dimensional. Hint: look at the dimensions of $\operatorname{Image} T$, $\operatorname{Image} T^2$, $\operatorname{Image} T^3$, $\dots$. Another way is to look at the dimensions of $\ker T$, $\ker T^2$, $\ker T^3$, $\dots$.