Math 219, Linear Algebra I — Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 4, for practice before Quiz 1 — do not hand in this problem set

Quiz 1: We will have the **written part** of Quiz 1 during class time on Wednesday, October 14, and the **oral part** on Thursday, October 15. Please **sign up** on Moodle for which timeslots you are available for the oral part. I will assign the timeslots once I know everybody's schedule.

Exercises from Corwin-Szczarba:
Section 4.1, exercises 1bcd, 2abeg, 4abc, 8, 9, 12.
Section 4.2, exercises 1ad, 2abgi.

Additional Exercises (also required):

Exercise A4.1: Let V be a vector space, and suppose $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} \in V$ satisfy the equation

$$2\vec{\mathbf{x}} + 3\vec{\mathbf{y}} + 4\vec{\mathbf{z}} = \vec{\mathbf{0}}.$$

Prove that $\operatorname{span}\{\vec{\mathbf{x}}, \vec{\mathbf{y}}\} = \operatorname{span}\{\vec{\mathbf{y}}, \vec{\mathbf{z}}\}$.

Exercise A4.2: Define a function $T : \mathcal{P}_2 \to \mathcal{P}_2$ by

$$T(f) = (1 - x^2)f'' + 3f' + 2f.$$

a) Show that T is a linear transformation.
b) Find a set of vectors in $\mathcal{P}_2$ whose span is $\operatorname{Image} T$. (Part (b) of Exercise A4.3 below may be useful.)
c) Find a set of vectors in $\mathcal{P}_2$ whose span is $\ker T$. Suggestion: if $f = ax^2 + bx + c$, what condition on the numbers a, b, c is equivalent to $f \in \ker T$? Solve for all possible (a, b, c) and reexpress in terms of the polynomial f.

Exercise A4.3: a) Let $T : V \to W$ be an **injective** linear transformation, and suppose that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k \in V$ are linearly independent vectors. Prove that the vectors $T(\vec{\mathbf{v}}_1), T(\vec{\mathbf{v}}_2), \ldots, T(\vec{\mathbf{v}}_k)$ are linearly independent.
b) Let $T : V \to W$ be a linear transformation (not necessarily injective), and suppose that $V = \operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$. Show that $\operatorname{Image} T = \operatorname{span}\{T(\vec{\mathbf{v}}_1), T(\vec{\mathbf{v}}_2), \ldots, T(\vec{\mathbf{v}}_n)\}$.
c) Based on part (b), what is an analogous statement to (a) for **surjective** linear transformations?

Exercise A4.4: Let $T : \mathbf{R}^5 \to \mathbf{R}^3$ be the linear transformation given by the matrix

$$A_T = \begin{pmatrix} 1 & 2 & -3 & 2 & -4 \\ 2 & 4 & -5 & 1 & -6 \\ 5 & 10 & -13 & 4 & -16 \end{pmatrix}.$$

Find (with proof) a basis for $\ker T$ and a basis for $\operatorname{Image} T$.

Further problems to look at:
Section 4.1, exercises 3, 5, 7, 11, 13.
Section 4.2, exercises 3, 5, 7, 8, 9, 11, 12, 13.