Math 341, Modules and Rings I – Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 3, due Tuesday, October 6 at 2pm via Moodle

Exercises from Jacobson, BA I:

Section 3.3, exercises 1, 2, 4 (this should read $\mathrm{Hom}_R(R, M) \cong M$), 7, 8.

Section 3.4, exercises 1, 2, 3, 5 (see remarks below).

Remark for exercise 3.4.1: there is a typo; it should say assume that the elements $f_1, \ldots f_m$ are a basis for $R^{(n)}$, not for $R^{(m)}$. The point is that the existence of a basis of $R^{(n)}$ with m elements implies that $R^{(n)}$ is isomorphic to $R^{(m)}$.

Remark for exercise 3.4.2: the notation $GL_n(R)$ means the group of invertible $n \times n$ matrices with entries in R; note R is not assumed commutative, so you cannot use determinants.

Additional Exercises (also required):

Exercise A3.1: Consider the $\mathbf{Q}[\lambda]$-module V given by $V = \mathbf{Q}^2$, where the action of λ is by the matrix $T = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$.

a) Describe for every $v \in V$ the annihilator ideal $\mathrm{Ann}\, v \subseteq \mathbf{Q}[\lambda]$. (Note that different v's may have different annihilators.)

b) Show that V is a cyclic $\mathbf{Q}[\lambda]$-module, and give an explicit isomorphism $V \cong \mathbf{Q}[\lambda]/I$ for a specific ideal I.

Exercise A3.2: Repeat Exercise A3.1, part (a) for the matrices A, B, C, D below (so $V = \mathbf{Q}^2$ or $V = \mathbf{Q}^3$). Determine in each case whether V is a cyclic $\mathbf{Q}[\lambda]$-module, in which case also solve part (b) of Exercise A3.1.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Possible suggestion for the cyclic cases: make the connection with the Chinese Remainder Theorem.

Look at, but do not hand in:

BA I, 3.3.9, 3.4.6 (note for 3.4.6: I suggest you look at an example with $R = \mathbf{Z}$, $n = 2$, $A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ to get a feel for what is going on. One direction of the "if and only if" is straightforward, but the other direction is difficult based on what you know at this point.)