

Math 341, Modules and Rings I – Fall 2020
Course website: <https://sites.aub.edu.lb/kmakdisi/>
Problem set 1, due Tuesday, September 22 at 2pm via Moodle

Exercises from Jacobson, BA I:

Section 2.5, exercise 4.

Section 2.7, exercises 10, 11 (do Exercise A1.1 before doing these). Hint for the first part of exercise 10: use $I_1 + I_2 = R$ to determine elements $e_1, e_2 \in R$ that satisfy

$$e_1 \equiv 1 \pmod{I_1}, \quad e_1 \equiv 0 \pmod{I_2}; \quad e_2 \equiv 0 \pmod{I_1}, \quad e_2 \equiv 1 \pmod{I_2}.$$

Then your desired a with $a \equiv a_1 \pmod{I_1}$ and $a \equiv a_2 \pmod{I_2}$ is an easy linear combination of e_1 and e_2 . It will help if you first work out the example $R = \mathbf{Z}$, $I_1 = \langle 7 \rangle$, $I_2 = \langle 10 \rangle$.

Section 2.15, exercise 5.

Additional Exercises (also required):

Exercise A1.1: Let R be a ring, not necessarily commutative, and let I, J be (two-sided) ideals of R .

a) The “ideal sum” $I + J$ is defined by

$$I + J = \{x \in R \mid \exists i \in I, j \in J \text{ such that } x = i + j\}.$$

The definition is usually abbreviated to $I + J = \{i + j \mid i \in I, j \in J\}$. Show that $I + J$ is an ideal containing I and J .

b) The “ideal product” IJ is defined by

$$IJ = \{x \in R \mid \exists \text{ finitely many } i_1, \dots, i_n \in I, j_1, \dots, j_n \in J \text{ such that } x = i_1 j_1 + \dots + i_n j_n\}.$$

Show that IJ is an ideal contained in $I \cap J$.

c) When R is a PID (if you like, let $R = \mathbf{Z}$), what do the ideal sum and ideal product mean? In other words, if $\langle a \rangle + \langle b \rangle = \langle s \rangle$, and $\langle a \rangle \langle b \rangle = \langle p \rangle$, what are s and p in terms of a and b ? What about the intersection $\langle a \rangle \cap \langle b \rangle = \langle i \rangle$?

d) Give an example of a ring R and ideals I, J with $IJ \neq I \cap J$.

e) Jacobson, BA I, exercises 2.5.2, 2.5.3.

Exercise A1.2: Let F be a field, and let $R = F[x, y]$ be the ring of polynomials in two variables. Show that the ideal $\langle x, y \rangle$ is not principal. Hence R is not a PID.

(Culture: $F[x, y]$ is still a UFD, as are all $F[x_1, x_2, \dots, x_n]$ or $\mathbf{Z}[x_1, x_2, \dots, x_n]$ or indeed all $D[x_1, x_2, \dots, x_n]$ for D a UFD. See Theorem 2.25 in Section 2.16 of BA I.)

Exercise A1.3: a) Let R be a PID, and let $I = \langle a \rangle$ be an ideal with a neither zero nor a unit. Consider a finite **strictly** ascending chain of ideals starting at I (“strictly” means consecutive ideals are never equal, and “finite” just means the chain does not go on forever):

$$(*) \quad I \subsetneq I_1 \subsetneq I_2 \subsetneq \dots \subsetneq I_n.$$

For a given a , what is the largest possible value of n ?

b) Now let $R = \mathbf{Q}[x, y]$. Show that if one starts with the ideal $I = \langle x \rangle$, there exist strictly ascending chains of ideals that can be as long as you like. So the n in (*) is not bounded above in this case. (This also implies that R is not a PID, but the proof in Exercise A1.2 is preferable because it is much more direct.)

Exercise A1.4: Let F be a field. Show that the ring $R = M_n(F)$ of $n \times n$ matrices with entries in F has only two ideals, namely $\{0\}$ and R . If the general case gives you trouble, do at least the case $n = 2$. (Culture: compare this result to BA I, exercises 2.5.4 and 2.5.8.)

Look at, but do not hand in:

BA I, 2.5.8, 2.6.5, 2.11.1, 2.11.2, 2.15.4, 2.15.11, 2.15.12, 2.15.13, 2.15.14, 2.15.15, 2.15.16.