Math 219, Linear Algebra I — Fall 2020
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 2, due Thursday, October 1 at 2pm via Moodle

Exercises from Corwin-Szczarba:

Section 2.4, exercises 13, 14, 15. (Note: problems 14 and 15 use the general definition of the span from Exercise A2.3 below. If you are stuck on problem 14 in its full generality, hand in a proof in the special case where S_1 and S_2 are finite sets.)

Section 2.5, exercises 1abcdh, 2, 6, 7, 9.

Additional Exercises (also required):

Exercise A2.1: Check whether or not each of the vectors below is in the appropriate span (whether in $\mathbf{R}^2$ or $\mathcal{P}_2$):

a) Is $\begin{pmatrix}2\\3\end{pmatrix} \in \operatorname{span}\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$?

b) Is $\begin{pmatrix}0\\0\end{pmatrix} \in \operatorname{span}\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\}$?

c) Is $\begin{pmatrix}1\\2\end{pmatrix} \in \operatorname{span}\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\}$?

d) Is $\begin{pmatrix}5\\5\end{pmatrix} \in \operatorname{span}\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\}$?

e) Is $x^2 \in \operatorname{span}\{2x^2+1, x+1, x^2+x\}$?

f) Is $x+1 \in \operatorname{span}\{x-1, x^2-1, x^2-3x+2, x^2-3x+3\}$?

g) Is $x+1 \in \operatorname{span}\{x-1, x^2-1, x^2-3x+2\}$?

h) Is $x^2-5x+4 \in \operatorname{span}\{x-1, x^2-1, x^2-3x+2\}$?

Exercise A2.2: Find a set of vectors that spans the following vector space:

$$V = \left\{ \begin{pmatrix}x\\y\\z\\t\\w\end{pmatrix} \in \mathbf{R}^5 \;\middle|\; \begin{array}{r} x+y+z+t \qquad = 0 \\ x+y+ \qquad 2t+w = 0 \\ 3x+5y+4z+6t+w = 0 \\ -2x+2y-2z+6t+4w = 0 \end{array} \right\}.$$

Exercise A2.3: (The general definition of the span.) Let V be a vector space, and let $S \subset V$ be any subset. Here S does not have to be a finite set (but it might be). So it is possible that S is the empty set $\emptyset$, and it is possible that S is an infinite set, i.e., containing infinitely many elements.

a) We define the span of S in this general situation as follows:

$$\vec{\mathbf{v}} \in \operatorname{span} S \iff \vec{\mathbf{v}} = \vec{\mathbf{0}} \text{ or } \vec{\mathbf{v}} = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + \cdots + a_k\vec{\mathbf{v}}_k, \text{ where } \vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k \in S$$

are finitely many vectors in S, and the $a_1, \ldots, a_k \in \mathbf{R}$. (So the vectors in $\operatorname{span} S$ are precisely those that can be written as a linear combination using **finitely many** vectors from S, possibly with repetition; note that the case $k = 0$, a linear combination using no vectors at all, can be interpreted as giving $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.) **Your question:** In $\mathcal{C}(\mathbf{R})$, what is the span of the set of functions $\{1, x, x^2, x^3, x^4, \ldots\}$?

b) In the special case when S is a nonempty finite set, with $S = \{\vec{\mathbf{A}}_1, \ldots, \vec{\mathbf{A}}_\ell\}$, show that this new definition of $\operatorname{span} S$ gives the same subspace as our previous definition from class. What about $\operatorname{span}\emptyset$? What is that?

c) Show that in all cases $\operatorname{span} S$ is a subspace of V and that $S \subset \operatorname{span} S$.

d) Show that for every subspace W of V, we have $S \subset W \iff \operatorname{span} S \subset W$.

Cultural note: parts (c) and (d) mean that $\operatorname{span} S$ is the smallest subspace that contains S. Compare to exercise 2.4.12 in the book.

Look at, but do not hand in, the following exercises:

Section 2.4, exercises 12, 16, 17.

Section 2.5, exercises 8, 11, 13.