Math 241, Introduction to Abstract Algebra – Fall 2019
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 9, due Wednesday, December 4 at the beginning of class

Exercises from Fraleigh:

Section 36, exercises 1, 2, 3, 4, 11, 12, 13. (For all of these, you may use Theorems 36.8, 36.10, and 36.11 as "black boxes" without needing to fully understand the proofs.)

Section 18, exercises 11, 12, 15, 20, 38.

Additional Exercises (also required):

Exercise A9.1: For each of S_4, S_5, A_4 and A_5, write down a representative x of each conjugacy class C_x, and find the order $|Z_x|$ of the centralizer Z_x of x as well as the cardinality $|C_x|$ of the conjugacy class in question.

As a check on your work: S_4 has 5 conjugacy classes, S_5 has 7 conjugacy classes, A_4 has 3 conjugacy classes, and A_5 has 5 conjugacy classes. Caution: elements of A_4 or A_5 may be conjugate in S_4 or S_5 without being conjugate in A_4 or A_5; in this case, those elements belong to distinct conjugacy classes in A_4 or A_5. Make sure that the sum of the cardinalities of your conjugacy classes equals the order of the group.

Exercise A9.2: Let G be a finite group, and let p be a prime factor of $|G|$. Give the "traditional" proof of Cauchy's theorem that G contains an element of order p, by completing the following sketch:

The proof is by induction on $|G|$, so assume the theorem is known for all groups H with smaller cardinality $|H| < |G|$. Let Z be the center of G. If p is a divisor of $|Z|$, you can use the simple Cauchy theorem on elements of Z. Otherwise, at least one conjugacy class C_x in G has cardinality that is neither 1, nor divisible by p (why?), and so the centralizer Z_x is a proper subgroup of G (why?) and p is now a factor of $|Z_x|$ (why?). Take $H = Z_x$ and complete the proof.

Look at, but do not hand in:

Section 36, exercises 16, 17, 18, 20, 21, 22.
Section 37, exercises 4, 5, 6.
Section 18, exercises 37, 41, 55, 56.

"Look At" Exercise L9.1: Look up the definitions of the regular dodecahedron and icosahedron.

a) Write down how many vertices, edges, and faces each of the two shapes has. How many edges emanate from each vertex?

b) Let G_d and G_i be the groups of rotations (elements of $SO(3)$) that are symmetries of the dodecahedron and icosahedron, respectively. How many elements does each group have? What would happen if we allowed reflections (i.e., elements of $O(3)$)?

c) For each of G_d and G_i acting on its regular solid, show that the stabilizer of a vertex, respectively of an edge, respectively of a face is cyclic, and find the order of each type of stabilizer. Verify that each orbit (i.e., of vertices, edges, and faces) has the correct cardinality, which is $|G|/|\text{stabilizer}|$.

d) Show that G_d and G_i are isomorphic to each other, and (challenge) that both are isomorphic to A_5.

"Look At" Exercise L9.2: For each prime factor p of $|G|$ below, find a p-Sylow subgroup:

$$\text{(i) } G = S_4, \qquad \text{(ii) } G = S_5, \qquad \text{(iii) } G = \mathbf{Z}_{100} \times \mathbf{Z}_{30}, \qquad \text{(iv) } G = GL_2(\mathbf{Z}_7).$$

Note: (iv) is challenging. At least try to find subgroups whose order p^n is as large as you can manage.