Math 241, Introduction to Abstract Algebra – Fall 2019
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 8, due Wednesday, November 13 at the beginning of class

Tentative dates: Quiz 2 on Monday, November 18 in class, and an extra lecture Wednesday, November 20 from 6:30-8:30 (location to be announced).

Exercises from Fraleigh:
Section 15, exercises 1, 3, 7, 8, 12, 35, 36.
Section 16, exercises 2, 3, 5, 6, 9, 13.

Look at, but do not hand in:
Section 15, exercises 28, 37, 39, 42.
Section 16, exercises 1, 4, 7, 10, 11, 12, 14–17, 19, 20.
Section 17, exercises 1, 2, 4, 5.

"Look At" Exercise L8.1: In this exercise, we consider the action of the group

$$SO(3) = \{A \in GL(3, \mathbf{R}) \mid A^t = A^{-1} \text{ and } \det A = +1\}$$

on $\mathbf{R}^3$ by $A \cdot v$ being the product of the matrix $A \in SO(3)$ with the (column) vector $v \in \mathbf{R}^3$.
Before proceeding, recall the length of a vector:

$$\text{If } v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3, \text{ then } \|v\| = \sqrt{x^2 + y^2 + z^2}.$$

Also convince yourself that every element $A \in O(3)$ (i.e., $A^t = A^{-1}$ without insisting on $\det A = 1$) can be described by taking two orthonormal bases $\{u_1, u_2, u_3\}$ and $\{u_1', u_2', u_3'\}$ of $\mathbf{R}^3$ and requiring that $Au_i = u_i'$ for $1 \leq i \leq 3$.

a) Show that v and w are in the same orbit of $SO(3)$ if and only if $\|v\| = \|w\|$. (Hint for the direction $\|v\| = \|w\| \Rightarrow v$ and w are in the same orbit: you may wish to first solve part (b) below.)

b) Find explicit $A, A', A'' \in SO(3)$ such that

$$A' \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \quad A'' \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}.$$

c) Show that if $v \neq (0, 0, 0)$, then $\operatorname{Stab} v$ is isomorphic to the (circle) group $\mathbf{R}/2\pi\mathbf{Z}$. (Hint: choose a suitable orthonormal basis $\{u_1, u_2, u_3\}$, and write u_1', u_2', u_3' in terms of the u_i.)

"Look At" Exercise L8.2: In this exercise, we let $G = GL(n, \mathbf{R})$ act on the set $X = M_n(\mathbf{R})$ of $n \times n$ matrices by conjugation:

$$g \bullet M = gMg^{-1} \quad \text{(matrix multiplication)}.$$

Recall that the characteristic polynomial of a matrix $M \in X$ is $\det(xI - M)$: for example, the characteristic polynomial of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\det \begin{pmatrix} x-1 & -2 \\ -3 & x-4 \end{pmatrix} = x^2 - 5x - 2$.

a) Show that all elements in the same orbit must have the same characteristic polynomial. We will call this the characteristic polynomial of the orbit.

b) If $n = 2$ and $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, show that the orbit of M consists exactly of those matrices whose characteristic polynomial is $(x-2)(x-3)$. What is the stabilizer of M?

c) If $n = 2$ and $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, show that the orbit of M is a **proper** subset of the matrices with characteristic polynomial $(x-1)^2$. What is the stabilizer of M? What other orbits have the same characteristic polynomial?

d) If $n = 2$, show that the matrices M whose characteristic polynomial is $x^2 + 1$ form a single orbit, which does not contain any diagonal matrices.

e) For any n, suppose $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}$ are **distinct** real numbers. How many orbits have characteristic polynomial $(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$?

f) The following matrices all have characteristic polynomial x^4; show that they all belong to different orbits. (Hint: if M and N are in the same orbit, then M and N have the same rank — why? — as do M^2 and N^2, etc.)

$$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$$