Math 241, Introduction to Abstract Algebra – Fall 2019
Course website: `https://sites.aub.edu.lb/kmakdisi/`
Problem set 4, due Wednesday, October 2 at the beginning of class

Exercises from Fraleigh:

Section 9, exercises 7, 10.

Note: for exercises 7 and 10 above, your job (unlike last time) is to find sign σ only, by three different methods: (i) by finding N_σ, (ii) by writing σ as a product of transpositions, or (iii) by using an expression of σ as a product of cycles.

Section 10, exercises 2, 4, 6, 7, 8, 9, 10, 11, 26, 27.
Note: exercises 6–11 are one unit.

Additional Exercises (also required):

Exercise A4.1: Show that for $n \geq 3$, the alternating group A_n is generated by the set of all 3-cycles. For example,

$$A_4 = \langle (123), (132), (124), (142), (134), (143), (234), (243) \rangle.$$

Incidentally, since $(132) = (123)^{-1}$ and so forth, we can also write

$$A_4 = \langle (123), (124), (134), (234) \rangle.$$

Exercise A4.2: Describe the elements of $D_n \cap A_n$. (The answer depends on n mod 4.)

Exercise A4.3: Using only the definition of the determinant of $A = (a_{ij})_{1 \leq i,j \leq n}$ in terms of permutations, and **without using any other properties of the determinant**, show that:

a) $\det A^t = \det A$, where A^t is the transpose of A. (Hint: first show $\operatorname{sign} \sigma^{-1} = \operatorname{sign} \sigma$.)

b) If A is upper triangular (for example, if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$), then $\det A = \prod_{i=1}^{n} a_{ii}$, the product of the entries on the diagonal.

c) If one exchanges two rows or columns of A, then the determinant changes sign. (Do either rows or columns, and then part (a) will give you the other case.) Hint: cosets of a subgroup $\{1, (ij)\}$ in S_n will play a role here. Try a couple of examples with $n \leq 4$ on scratch paper before figuring out the general proof.

d) If any two rows or columns of A are equal, then $\det A = 0$.

Reminder: $\det A = \sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$.

Look at, but do not hand in:

Section 9, exercises 31, 32.

"Look at" Exercise L4.1: Let $\sigma \in S_n$. Consider an **adjacent** transposition $(i\ i{+}1)$, and let $\tau = \sigma(i\ i{+}1)$. Show that $N_\tau = N_\sigma + 1$ or $N_\sigma - 1$, depending on a condition that you must determine. Use this to show that N_σ is the smallest number of adjacent transpositions whose product can give σ.

It may help you to start by working out some examples, such as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & \mathbf{3} & \mathbf{1} & 4 & 2 \end{pmatrix}, \qquad \tau = \sigma(23) = \text{(why?)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & \mathbf{1} & \mathbf{3} & 4 & 2 \end{pmatrix}$$

and various others.