Math 201 — Calculus and Analytic Geometry III
Handout on Taylor's theorem

The purpose of this handout is to sketch a proof of "Taylor's theorem", which gives a formula for the difference between a function and its nth Taylor polynomial. The formula reads:

$$f(b) = P_n(b)+R_n(b) = \left[f(a)+f'(a)(b-a)+\frac{f''(a)}{2}(b-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(b-a)^n\right]+\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

for some c between a and b. We will only sketch the proof in case $a < b$; the case of $b < a$ is quite similar, and the case $b = a$ is trivial. For this proof, we will also assume the function f is a bit nicer than required in the book. Namely, we will assume that $f^{(n+1)}(x)$ not only exists on the open interval (a, b), but also exists and is **continuous** on the **closed** interval $[a, b]$.

1. Inequalities involving integrals. Let $\ell(x)$, $g(x)$, and $u(x)$ be continuous functions for $x \in [a, b]$ (i.e., for $a \leq x \leq b$), and assume that

$$\ell(x) \leq g(x) \leq u(x), \text{ for all } x \in [a, b].$$

Think of $\ell(x)$ as a lower bound for $g(x)$, and $u(x)$ as an upper bound for $g(x)$. Then we can conclude the following inequality:

$$\int_{t=a}^{x} \ell(t)\, dt \leq \int_{t=a}^{x} g(t)\, dt \leq \int_{t=a}^{x} u(t)\, dt, \text{ for all } x \in [a, b].$$

Draw a picture to see why this is the case. (This is where we need $a < b$; otherwise, we would have to change the direction of the inequalities, but the rest of the proof is otherwise identical.)

2. Taylor's theorem for $n = 0$, i.e., the Mean Value Theorem. Assume that $f(x)$ is differentiable and $f'(x)$ is continuous for all $x \in [a, b]$. Let M, m be the maximum and minimum values of $f'(x)$ for $x \in [a, b]$. Thus, for $a \leq x \leq b$ we obtain

$$m \leq f'(x) \leq M \Rightarrow \int_{t=a}^{x} m\, dt \leq \int_{t=a}^{x} f'(t)\, dt \leq \int_{t=a}^{x} M\, dt \Rightarrow m(x-a) \leq f(x) - f(a) \leq M(x-a).$$

We can rewrite this as

$$f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a), \text{ for } x \in [a, b].$$

If we take $x = b$, this means that we can write $f(b) = f(a) + d(b-a)$ for some choice of d between m and M. Now m and M are the minimum and maximum values of $f'(x)$ for $x \in [a, b]$. So by the intermediate value theorem, it is possible to find at least one choice of $c \in [a, b]$ such that $f'(c) = d$, and we obtain the mean value theorem:

$$f(b) = f(a) + f'(c)(b-a), \text{ for some choice of } c \in [a, b].$$

3. The second derivative. Now assume that f' and f'' both exist and are continuous. Write m_2 and M_2 for the minimum and maximum values of $f''(x)$ for $x \in [a, b]$. We now apply the result of section 2 to the function f' instead of f, and obtain

$$f'(a) + m_2(x-a) \leq f'(x) \leq f'(a) + M_2(x-a), \text{ for } x \in [a, b].$$

If we integrate these inequalities, we obtain

$$\int_{t=a}^{x} f'(a)\, dt + m_2 \int_{t=a}^{x} (t-a)\, dt \leq \int_{t=a}^{x} f'(t)\, dt \leq \int_{t=a}^{x} f'(a)\, dt + M_2 \int_{t=a}^{x} (t-a)\, dt, \text{ for } x \in [a, b].$$

Now $f'(a)$ does not depend on t, so $\int_{t=a}^{x} f'(a)\,dt = f'(a)(x-a)$. On the other hand,

$$\int_{t=a}^{x} (t-a)\,dt = \left[\frac{(t-a)^2}{2}\right]_{t=a}^{x} = \frac{(x-a)^2}{2} - 0 = \frac{(x-a)^2}{2}.$$

Substituting this into the above inequalities gives

$$f'(a)(x-a) + m_2\frac{(x-a)^2}{2} \le f(x) - f(a) \le f'(a)(x-a) + M_2\frac{(x-a)^2}{2}.$$

We can rewrite this as

$$f(a) + f'(a)(x-a) + m_2\frac{(x-a)^2}{2} \le f(x) \le f(a) + f'(a)(x-a) + M_2\frac{(x-a)^2}{2}, \text{ for } x \in [a,b].$$

Once again, if we take $x = b$, we conclude that

$$f(b) = f(a) + f'(a)(b-a) + d_2\frac{(b-a)^2}{2}, \text{ for some } d_2 \text{ with } m_2 \le d_2 \le M_2.$$

The intermediate value theorem for f'' then implies that we can write $d_2 = f''(c_2)$, for at least one choice of $c_2 \in [a,b]$. This gives us Taylor's theorem for $n = 1$.

4. Higher derivatives. The general case follows by induction on n, following the pattern that we have started. It is left as an exercise for you to show that if we know the theorem for $n = k$, then we know it for $n = k+1$. The details are quite simple, but the notation takes up some space to write down for general k. Instead, here is a sketch of how one deduces the theorem for $n = 2$ from the theorem for $n = 1$:

Let m_3 and M_3 be the minimum and maximum values of $f'''(x)$ for $x \in [a,b]$. We put $g = f'$ instead of f the result of the previous section. Since $g'' = f'''$, the minimum and maximum values of g'' are m_3 and M_3, so we obtain

$$f'(a) + f''(a)(x-a) + m_3\frac{(x-a)^2}{2} \le f'(x) = g(x) \le f'(a) + f''(a)(x-a) + M_3\frac{(x-a)^2}{2}, \text{ for } x \in [a,b].$$

Just like before, we integrate the inequalities, using the facts that $\int_{t=a}^{x} f'(a)\,dt = f'(a)(x-a)$,

$$\int_{t=a}^{x} f''(a)(t-a)\,dt = f''(a)\cdot\frac{(x-a)^2}{2}, \qquad \int_{t=a}^{x} \frac{(t-a)^2}{2}\,dt = \left[\frac{(t-a)^3}{3!}\right]_{t=a}^{x} = \frac{(x-a)^3}{3!}.$$

We obtain that for $x \in [a,b]$,

$$f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + m_3\frac{(x-a)^3}{3!} \le f(x) - f(a) \le f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + M_3\frac{(x-a)^3}{3!}.$$

We rearrange as usual to obtain a sandwich for $f(x)$ between the two quantities:

$$P_2(x) + m_3\frac{(x-a)^3}{3!} \le f(x) \le P_2(x) + M_3\frac{(x-a)^3}{3!}, \text{ for } x \in [a,b].$$

Again, we express

$$f(b) = P_2(b) + d_3\frac{(b-a)^3}{3!}, \text{ for some } d_3 \text{ with } m_3 \le d_3 \le M_3.$$

By the intermediate value theorem for f''', we know as usual that

$$d_3 = f'''(c_3), \text{ for some choice of } c_3 \text{ with } a \le c_3 \le b.$$