Chart of orders of growth

	logarithmic growth	polynomial growth				exponential growth		factorial growth
n	$\ln n$	$n^{0.1}$	$100n$	n^{2}	n^{100}	1.01^{n}	2^{n}	$n!$
1	0.00000	1.00000	100	1	1	1.01000	2	1
2	0.69315	1.07177	200	4	1.27E+30	1.02010	4	2
3	1.09861	1.11612	300	9	5.15E+47	1.03030	8	6
4	1.38629	1.14870	400	16	1.61E+60	1.04060	16	24
5	1.60944	1.17462	500	25	7.89E+69	1.05101	32	120
6	1.79176	1.19623	600	36	6.53E+77	1.06152	64	720
7	1.94591	1.21481	700	49	3.23E+84	1.07214	128	5040
8	2.07944	1.23114	800	64	2.04E+90	1.08286	256	40320
9	2.19722	1.24573	900	81	2.66E+95	1.09369	512	362880
10	2.30259	1.25893	1000	100	1.00E+100	1.10462	1024	3628800
20	2.99573	1.34928	2000	400	1.27E+130	1.22019	1048576	2.43E+18
50	3.91202	1.47876	5000	2500	7.89E+169	1.64463	1.13E+15	3.04E+64
100	4.60517	1.58489	10000	10000	1E+200	2.70481	1.27E+30	9.33E+157
1000	6.90776	1.99526	100000	1000000	1E+300	20959.16	1.07E+301	>1E+2567
10000	9.21034	2.51189	1000000	100000000	1E+500	1.64E+43	>1E+3010	>1E+35659
100000	11.51293	3.16228	10000000	1E+10	1E+600	>1E+432	>1E+30102	>1E+456573
1E+100	230.2585	1E+10	1E+102	1E+200	1E+10000	>1E(10^97)	>1E(10^99)	>1E(10^101)

Math 201 — Calculus and Analytic Geometry III
Handout on orders of growth

The purpose of this handout is to give an idea of how one proves that logarithmic growth is much slower than polynomial growth, which is in turn much slower than exponential growth, which in turn is much smaller than factorial growth. As an unrelated bonus, we include a proof of the formula $\lim_{n\to\infty}\left(1+\dfrac{c}{n}\right)^n = e^c$.

Step 1. Linear growth is no worse than exponential growth.

Fix $a > 0$. We wish to study $f(x) = \dfrac{x}{e^{ax}} = xe^{-ax}$. We are not yet ready to show that $\lim_{x\to+\infty} f(x) = 0$. Instead, we will prove the weaker result:

$$\text{For } x \geq 0, \quad f(x) = xe^{-ax} \text{ is bounded.}$$

To do this, we find the derivative $f'(x) = (1)(e^{-ax}) + (x)(-ae^{-ax}) = (1-ax)e^{-ax}$. Now $e^{-ax} > 0$, so $f'(x)$ has the same sign as $(1-ax)$. This means (check!) that $f'(x) > 0$ for $0 < x < 1/a$, and $f'(x) < 0$ for $x > 1/a$. Thus the table of variation for f looks like:

x	0		$1/a$			$+\infty$
$f'(x)$	1	$+$	0	$-$	from here on, $f' < 0$	
$f(x)$	0	$\nearrow$	$f(1/a)$	$\searrow$	$f(x)$ stays positive (since x and $e^{-ax} > 0$)	

The point of this is that for $x \geq 0$, the values of $f(x) = xe^{-ax}$ are always between 0 and the maximum value $f(1/a) = (1/a)e^{-a(1/a)} = 1/ea$. It does not really matter what this maximum value is; we can just call it $C_a = 1/ea = f(1/a)$ since it is a constant that does not depend on x (even though it depends on a). We conclude:

$(*)$ Fix $a > 0$. Then for all $x \geq 0$, $0 \leq xe^{-ax} \leq C_a$, or equivalently $0 \leq x \leq C_a e^{ax}$.

Step 2. Linear growth is much smaller than exponential growth. Let us show for example that $\lim_{x\to+\infty} \dfrac{x}{e^x} = 0$. In other words, for sufficiently large x, x is much smaller than e^x. The proof is as follows: we use inequality $(*)$ above with $a = 0.01$ to conclude that

$$\text{For } x \geq 0, \quad 0 \leq \frac{x}{e^x} = \left(xe^{-0.01x}\right)e^{-0.99x} < C_{0.01}e^{-0.99x}.$$

But $C_{0.01}e^{-0.99x} \to 0$ when $x \to +\infty$, so x/e^x is sandwiched between 0 and something that decays exponentially. Thus $x/e^x \to 0$ when $x \to +\infty$ by the sandwich theorem.

Remark: if we had used $(*)$ with $a = 1$, we would have deduced only that $0 \leq x/e^x \leq C_1$ for all $x \geq 0$. This would have shown that the ratio $x/e^x = xe^{-x}$ was bounded, but it would not have shown that $xe^{-x} \to 0$. This is why we had to decompose the exponential decay into $e^{-x} = e^{-0.01x}e^{-0.99x}$. The first factor $e^{-0.01x}$ "neutralizes" the growth of x because of $(*)$, while the second factor $e^{-0.99x}$ causes the decay. We can take other values than $a = 0.01$; any $a < 1$ would also work in this proof.

Step 3. Polynomial growth is much smaller than exponential growth. Here is a typical example: we want to show that $\lim_{x\to+\infty} \dfrac{x^{100}}{e^{0.001x}} = 0$. Thus the fast growth of x^{100} is small when compared to the extremely fast growth of $e^{0.001x}$. The trick will be to compare x to e^{ax} for an extremely small a. That way, we have

$$x^{100}e^{-0.001x} \leq (C_a e^{ax})^{100}e^{-0.001x} = (C_a)^{100}e^{100ax}e^{-0.001x} = (C_a)^{100}e^{(100a-0.001)x}.$$

Here we need to choose a so small that $100a - 0.001 < 0$. For example, take $a = 10^{-6}$ so that $100(10^{-6}) - 0.001 = 0.0001 - 0.001 = -0.0009$. Then we obtain that for $x \geq 0$, we have

$$0 \leq x^{100}e^{-0.001x} \leq (C_{10^{-6}})^{100}e^{-0.0009x}$$

and we have again sandwiched the ratio $\dfrac{x^{100}}{e^{0.001x}}$ between 0 and an exponential decay of the form $Ae^{-0.0009x}$ with the constant $A = (C_{10^{-6}})^{100}$. (The exact value of A does not matter for this argument.) From the fact that $0 \leq \dfrac{x^{100}}{e^{0.001x}} \leq Ae^{-0.0009x}$, we conclude by the sandwich theorem that $\dfrac{x^{100}}{e^{0.001x}} \to 0$. A similar argument works for any ratio of the form $\dfrac{x^c}{e^{dx}}$, where $c, d > 0$.

Step 4. Logarithmic growth is much smaller than polynomial growth. One way to prove this is similar to what we did in Steps 1–3. Replace the function xe^{-ax} in Step 1 with the new function $f(x) = x^{-a}\ln x$ for $x \geq 1$. The table of variations of this new function is similar to the one in Step 1, and we conclude that $x^{-a}\ln x$ is bounded by some constant C'_a. It is easy to adapt the ideas of Steps 2 and 3 as well, to show results such as $\dfrac{\ln x}{x} \to 0$ and $\dfrac{(\ln x)^{100}}{x^{0.001}} \to 0$ as $x \to +\infty$. You should fill in the details of this approach.

Another way is to write $x = e^t$ from the beginning, so $t = \ln x$. This transforms the problem of comparing $(\ln x)^c$ and x^d to the problem of comparing t^c to e^{dt}, which we have already solved. (We have $x \to +\infty \iff t \to +\infty$.) This incidentally shows that the constant C'_a above is the same as C_a from Step 1, because the function $x^{-a}\ln x$ becomes $e^{-at}t$ which is the function we originally considered.

Step 5. Exponential growth is much smaller than factorial growth. Let us show for example that $\lim_{n\to+\infty} \dfrac{100^n}{n!} = 0$. A similar proof works for $\lim \dfrac{A^n}{n!}$ for any $A > 1$. Let $n \geq 201$. Then

$$0 \leq \frac{100^n}{n!} = \frac{100 \cdot 100 \cdot 100 \cdots 100}{(1)\cdot(2)\cdot(3)\cdots(199)} \cdot \left(\frac{100}{200}\right) \cdot \left(\frac{100}{201}\right) \cdots \left(\frac{100}{n}\right).$$

Now each of the fractions $(100/200)$, $(100/201)$, $\ldots$, $(100/n)$ is $\leq 1/2$, because the denominator is ≥ 200. This allows us to deduce that

$$0 \leq \frac{100^n}{n!} \leq \frac{100 \cdot 100 \cdot 100 \cdots 100}{(1)\cdot(2)\cdot(3)\cdots(199)} \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) \cdots \left(\frac{1}{2}\right) = \frac{100^{199}}{199!} \cdot \left(\frac{1}{2}\right)^{n-199} = \frac{A}{2^n},$$

where the factor $A = \dfrac{100^{199} \cdot 2^{199}}{199!}$ does not depend on n. Since $A/2^n \to 0$, we can use the sandwich theorem to deduce that $100^n/n! \to 0$, as desired.

Bonus: proof that $\lim_{n\to\infty}\left(1 + \dfrac{c}{n}\right)^n = e^c$. We first prove a preliminary result:

$$\lim_{x\to 0} \frac{\ln(1+cx)}{x} = c.$$

This result can be proved by L'Hôpital's rule (look it up in the book if you do not know it) or by a direct argument. The direct argument goes as follows: let $f(x) = \ln(1+cx)$. Then $f(0) = \ln 1 = 0$ and $f'(0) = c$ (because $f'(x) = c/(1+cx)$ due to the chain rule). By the definition of the derivative,

$$c = f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 0}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{\ln(1+cx)}{x}, \quad \text{as desired.}$$

We can now give the proof of our identity. Write $a_n = \left(1 + \dfrac{c}{n}\right)^n$ and $b_n = \ln a_n = n\ln(1 + c/n) = \ln(1+c/n)/(1/n)$. As $n \to +\infty$, we have $1/n \to 0$. Hence by the above identity $b_n \to c$. Since $a_n = e^{b_n}$ and e^x is a continuous function, we conclude that $a_n \to e^c$ as desired.