ON THE IWAHORI-HECKE ALGEBRA OF A p-ADIC GROUP

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1. Introduction

Let G be a split connected reductive algebraic group over a non-archimedean local field F of characteristic 0 with ring of integers $\mathcal{O}$, absolute value $|\cdot|$, and uniformizer π. Let $\mathcal{S}(G)$ be the category of smooth representations of G. Bernstein [BeD] has given a decomposition of $\mathcal{S}(G)$ into full subcategories

$$\mathcal{S}(G) = \sqcup \mathcal{S}_i(G)$$

with the subcategories indexed by pairs (M, σ) with M a Levi subgroup, and σ an irreducible supercuspidal representation of M, under a certain equivalence relation ("association") to be explained below.

In [BR], based on unpublished work of Bernstein, the authors construct a projective generator (to be defined below) for each subcategory $\mathcal{S}_i(G)$. For a given subcategory, the generators are not unique, and in fact the method in general gives a way to produce a finite number of non-isomorphic generators. Nevertheless, there is, up to isomorphism, a canonical choice of generator, Π_i, for each $\mathcal{S}_i(G)$, and this leads naturally to the equivalence of categories

$${}^r\mathfrak{Mod}\, \mathrm{End}_{\mathcal{S}(G)}(\Pi_i) \equiv \mathcal{S}_i(G),$$

where ${}^r\mathfrak{Mod}\, R$ is the category of right modules over a ring R.

This paper concerns analyzing the algebra $\mathrm{End}_{\mathcal{S}(G)}(\Pi_i)$ in the case where $\mathcal{S}_i(G)$ is the familiar unramified principal series. We note, however, that the results in [BR] are not needed for this special case. Our study works once the structure theory of the Iwahori-Hecke algebra in section 3 along with the classical "Borel result" on the unramified principal series are known. The latter says that this category is equivalent to the category of representations of the Iwahori-Hecke algebra (this was done first for admissible representations in [Bo] and then generally for smooth representations by Matsumoto [Ma]).

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To discuss this in more detail, let I be an Iwahori subgroup of G. The Iwahori-Hecke algebra is

$$\mathcal{H}_I = C_c^\infty(I\backslash G/I),$$

the convolution algebra of smooth compactly supported functions on G bi-invariant under I with respect to a Haar measure on G such that the measure of I is one. The main result of this paper is the construction of a natural isomorphism

$$\mathcal{H}_I \xrightarrow{\sim} \operatorname{End}_{\mathcal{S}(G)}(\Pi), \tag{1.1}$$

where Π is a certain projective generator for the unramified principal series. This isomorphism gives an essentially new point of view for studying the Iwahori-Hecke algebra. The main advantage of this point of view is that it is available equally to all the pieces of the Bernstein center.

The work of [KL] and [BK] shows that Iwahori-Hecke algebras play a central role in the representation theory of p-adic groups. In [KL] Kazhdan and Lusztig classify the irreducible elements of the unramified principal series by finding a geometric construction of the corresponding Iwahori-Hecke algebra and classifying its representations. This classification indeed resolves the Deligne-Langlands conjecture for this special case.

In [BK] Bushnell and Kutzko analyze the irreducible representations of $GL_n(F)$, and show that a key role is played by Iwahori-Hecke algebras.

2. Review of the Bernstein center

The Bernstein center plays a dominant role in the general theory that follows, so we review the fundamental concepts here. All of the definitions and facts stated in this section may be found, with proofs, in [BR]. Some of the facts may also be found in [BeD].

In this section, $\underline{G}$ is an arbitrary connected reductive linear algebraic group over a local field F, and $G = \underline{G}(F)$ is its F-points. Moreover, $\underline{M}$ is a Levi subgroup of $\underline{G}$ and $M = \underline{M}(F)$ is its F-points. We allow the case $\underline{M} = \underline{G}$; of particular interest is the case where $\underline{G}$ is split and $\underline{M}$ is a split maximal torus.

2.1. **Notation.** We adopt the following notation throughout this paper. Let M be a Levi subgroup of G, and $P = MN$ a Levi decomposition of a parabolic subgroup P. Then $i_{G,M}$ represents parabolic induction from M to G with respect to P, and $r_{M,G}$ represents Jacquet restriction with respect to P.

We will also need to employ ordinary compact induction. If $H \subset G$ is a closed subgroup, and ψ is a representation of H, then let $c\text{-}ind_H^G \psi$ represent the representation of G obtained by compactly inducing ψ from H to G.

2.2. **Cuspidal data I.** Let σ be an irreducible supercuspidal representation of M. The pair (M, σ) is called a *cuspidal datum.* Two cuspidal data:

$$(M, \sigma) \quad \text{and} \quad (M', \sigma')$$

are called *associate* if there exists $g \in G$ yielding isomorphisms

$$\operatorname{Ad} g : M \tilde{\to} M', \quad \text{and} \quad {}^g\sigma' \tilde{\to} \sigma,$$

where

$${}^g\sigma'(m) = \sigma(g^{-1}mg).$$

The property of being associate is an equivalence relation; its equivalence classes are called *association classes.*

2.3. **Cuspidal data II.** Let $\Omega(G)$ be the set of association classes of cuspidal data of G. The following theorem explains the significance of the set $\Omega(G)$.

Theorem 2.1.
(1) *Let κ be a smooth irreducible representation of G. Then there exists a Levi subgroup M, an irreducible supercuspidal representation ρ of M, and a surjection $r_{M,G}(\kappa) \twoheadrightarrow \rho$. By Frobenius reciprocity, this yields an injection*

$$\kappa \hookrightarrow i_{G,M}(\rho).$$

(2) *Let κ be a smooth irreducible representation of G. Then up to being associate, there is a unique cuspidal datum (M, ρ) (where M is a Levi subgroup of G) such that $\kappa \hookrightarrow i_{G,M}(\rho)$.*

Note that the first part of the theorem is conceptually easy to see. By transitivity of induction there exists a minimal Levi subgroup M such that $r_{M,G}(\kappa) \neq 0$. It remains to see that $r_{M,G}(\kappa)$ has an irreducible quotient, but this follows from the (non-trivial) fact that $r_{M,G}$ takes finitely generated representations to finitely generated representations.

2.4. **Unramified characters.** Let M be a Levi subgroup of G. An unramified character of M is a character which is the composition of a rational character $M \to F^\times$ and an unramified character $F^\times \to \mathbb{C}^\times$. (The latter is a character which is trivial on $\mathcal{O}^\times$.)

Let M_0 be the intersection of the kernels of all unramified characters of M. It is known that M_0 is generated by the union of all compact subgroups of M, and M/M_0 is a finitely generated $\mathbb{Z}$-lattice.

Let $\Psi(M)$ be the abelian group of unramified characters of M. We record the following fact, which will be useful later:

$$\mathbb{C}[\Psi(M)] \simeq \mathbb{C}[M/M_0] \simeq c\text{-}ind_{M_0}^{M}\mathbf{1}, \tag{2.2}$$

where $\mathbb{C}[M/M_0]$ is the group ring of finitely supported complex functions on M/M_0; $\mathbb{C}[\Psi(M)]$ is the coordinate ring of $\Psi(M)$ (recall that $\Psi(M)$ is a complex algebraic torus); and $\mathbf{1}$ is the trivial one-dimensional representation. The space $\mathbb{C}[M/M_0]$ with the action of M by translation can be thought of as the *universal unramified character* of M.

Given any smooth representation ρ of G and unramified character χ of G, we write $\chi\rho$ for the representation $\chi \otimes \rho$, and note that $\chi\rho$ is also smooth.

2.5. **Cuspidal components I.** It is known that if σ is an irreducible supercuspidal representation of G, then $\chi\sigma$ is also irreducible supercuspidal. We write $\mathrm{Irr}_c G$ for the set of isomorphism classes of irreducible supercuspidal representations of G. A *cuspidal component* is an orbit, D, of $\Psi(G)$ in the set $\mathrm{Irr}_c G$, where $\Psi(G)$ acts on $\mathrm{Irr}_c G$ by tensor product. Thus, a cuspidal component D has the form

$$D = \{\chi\sigma \mid \chi \in \Psi(G)\}.$$

The most important fact we need is that a cuspidal component *splits* the category $\mathcal{S}(G)$; that is, every $V \in \mathcal{S}(G)$ may be written as a direct sum of subrepresentations $V(D) \oplus V(D^{\perp})$ such that:

(1) Every irreducible subquotient of $V(D)$ is contained in D, and
(2) No irreducible subquotient of $V(D^{\perp})$ is contained in D.

2.6. **Projective generators I.** Let D be a cuspidal component for $\mathcal{S}(G)$, and let $\sigma \in D$. Define the full subcategory $\mathcal{S}(D)$ of $\mathcal{S}(G)$ as the subcategory of representations in $\mathcal{S}(G)$, all of whose irreducible subquotients are in D. Put another way, $\mathcal{S}(D)$ is the set of G-modules such that $V = V(D)$.

Consider the object

$$\Pi(D) = c\text{-}ind_{G_0}^{G}\sigma.$$

Note that $\Pi(D)$ depends only on D, and not on the particular choice of $\sigma \in D$. Then we have the following theorem, due to Bernstein:

Theorem 2.3. (1) $\Pi(D) \in \mathcal{S}(D)$,
(2) $\Pi(D)$ *is a projective object in* $\mathcal{S}(D)$,
(3) $\Pi(D)$ *is finitely generated, and*
(4) *The functor* $X \mapsto \mathrm{Hom}(\Pi(D), X)$ *is faithful for* $X \in \mathcal{S}(D)$.

Corollary 2.4. *The functor* $X \mapsto \operatorname{Hom}(\Pi(D), X)$ *induces an equivalence of categories*

$$\mathcal{S}(D) \equiv {}^{r}\mathfrak{Mod}\,\operatorname{End}_{\mathcal{S}(D)}(\Pi(D)).$$

The object $\Pi(D)$ is called a *projective generator* for the category $\mathcal{S}(D)$.

2.7. **Bernstein's algebra.** We pause to indicate Bernstein's approach to studying the algebra $\Lambda(D) = \operatorname{End}(\Pi(D))$. The idea is to view $\Pi(D)$ as a family of representations of G, parametrized by the elements of $\Psi(G)$. In fact, if we write $\mathcal{F} = \mathbb{C}[G/G_0] \simeq \mathbb{C}[\Psi(G)]$, then there is a (non-canonical) isomorphism

$$\mathcal{F} \otimes \sigma \xrightarrow{\sim} \Pi(D).$$

A character $\psi \in \Psi(G)$ gives rise to an algebra homomorphism $\mathcal{F} \to \mathbb{C}$, whose kernel is a maximal ideal m_ψ. (m_ψ is the "point" $\psi \in \Psi(G) = \operatorname{maxspec}\mathcal{F}$.) As ψ varies, $\Pi(D)_\psi = \Pi(D)/m_\psi\Pi(D)$ ranges over the representations belonging to the component D.

2.8. **Cuspidal components II.** Let M be a Levi subgroup of G. A cuspidal component of M is an orbit of the action of $\Psi(M)$ in the set $\operatorname{Irr}_c M$. By abuse of terminology, when we say "cuspidal component" without reference to a particular Levi subgroup, we mean first a cuspidal datum (M, σ) for G, and second the orbit of σ under $\Psi(M)$. Under this terminology the term *associate* cuspidal components has a well-defined meaning, so we also regard association classes of cuspidal components as subsets of $\Omega(G)$. We call these subsets *components* of $\Omega(G)$.

We now show how a component of $\Omega(G)$ gives rise to a full subcategory of $\mathcal{S}(G)$. Let $\operatorname{Irr} G$ denote the set of irreducible smooth representations of G. Define the map

$$\operatorname{pr}: \operatorname{Irr} G \to \Omega(G), \tag{2.5}$$

associating to an irreducible smooth representation κ the unique (up to being associate) cuspidal datum (M, ρ) such that $\kappa \hookrightarrow i_{G,M}\rho$. It is known that this map is finite-to-one and onto.

Given a component $\Omega \subset \Omega(G)$, we write $\operatorname{Irr}_\Omega G$ for the inverse image $\operatorname{pr}^{-1}\Omega$. Concretely, the set $\operatorname{Irr}_\Omega G$ consists of those irreducible smooth representations κ of G such that there is an element $(M, \rho) \in \Omega$ with $\kappa \hookrightarrow i_{G,M}(\rho)$.

We now state the Bernstein decomposition theorem. Let $V \in \mathcal{S}(G)$, and for a given component Ω of $\Omega(G)$ write $V(\Omega)$ for the maximal subrepresentation of V all of whose irreducible subquotients are in $\operatorname{Irr}_\Omega G$. Write $\mathcal{S}(\Omega)$ for the category of representations V such that $V = V(\Omega)$.

Theorem 2.6. *We have the following decomposition of the category* $\mathcal{S}(G)$

$$\mathcal{S}(G) = \prod_{\Omega} \mathcal{S}(\Omega),$$

where the product runs over all components of $\Omega(G)$*. In concrete terms this says that any smooth representation* V *of* G *may be written as a direct sum*

$$V = \oplus_{\Omega} V(\Omega).$$

2.9. **Projective Generators II.** Fix a component Ω of $\Omega(G)$. We define a projective generator $\Pi(\Omega)$ for $\mathcal{S}(\Omega)$. Pick (up to being associate) any cuspidal datum $(M, \sigma) \in \Omega$, and let D be the orbit of σ under $\Psi(M)$. Let $\overline{\Pi} = \Pi(D) = c\text{-}ind_{M_0}^{M}\sigma$ be the projective generator for the category $\mathcal{S}(D)$, and define $\Pi(\Omega) = i_{G,M}(\bar{\Pi})$. We may think of $\Pi(\Omega)$ as a "universal" parabolic induction of a twist of σ by an unramified character of M. Then we have the following crucial theorem, due to Bernstein:

Theorem 2.7. *The object* $\Pi(\Omega)$ *is a projective generator for the category* $\mathcal{S}(\Omega)$*, and therefore*

$$\mathcal{S}(\Omega) \equiv {}^{r}\mathfrak{Mod}\, \mathrm{End}_{\mathcal{S}(\Omega)}(\Pi(\Omega)).$$

This theorem, combined with the Bernstein decomposition (theorem 2.6) says that the entire category of smooth representations of G can be reduced to the study of modules over certain endomorphism algebras.

3. Structure theory of split p-adic groups

In this section we outline the necessary structure theory for split p-adic groups which we shall need.

The group G always has a *generalized Tits system* (see [Iw]). This is a triple $(G, I, \mathcal{N})$ satisfying the following properties:

(1) $T_0 = I \cap \mathcal{N}$ is a normal subgroup of $\mathcal{N}$.
(2) There exists a Coxeter group W' generated by a set of simple reflections S, a group Ω, and an isomorphism $\mathcal{N}/T_0 \simeq W' \rtimes \Omega$. See section 4 for a description of Ω; if G is semisimple then Ω is finite.
(3) The following two properties hold for the elements of S:
   (a) For $w \in W' \rtimes \Omega$ and $s \in S$, we have $wIs \subset IwsI \cup IwI$.
   (b) For all $s \in S$ we have $sIs^{-1} \neq I$.
(4) For $\rho \in \Omega$ we have $\rho S \rho^{-1} = S$, and $\rho I \rho^{-1} = I$.
(5) G is generated by I and $\mathcal{N}$.

Write $W = W' \rtimes \Omega$. We call this the *affine Weyl group* of G (as is clear from above, this is a Coxeter group only when $\Omega = \{1\}$).

There exists a Borel subgroup B with unipotent radical U. Let U^- be the unipotent radical of the Borel subgroup opposite to B, and let T be a maximal split torus of G. Let K be a special maximal compact subgroup of G containing T_0. We can (and will) choose (B, T, K) so that the following statements hold:

(1) Write $\Lambda = X_*(T)$ for the *coweight lattice* of G. We have $T \supset T_0$, and there is a natural isomorphism $\Lambda \tilde{\rightarrow} T/T_0$, $\lambda \mapsto t_\lambda$, where $t_\lambda = \lambda(\pi)$.

(2) (Compatibility of B and I) We have $I \cap U = K \cap U$. This means, for example, that there is an $\mathcal{O}$-structure on G such that I is obtained as the set of $\mathcal{O}$-points which reduce mod π to the reduction of B mod π.

(3) (Iwahori Factorization) Write $I^- = I \cap U^-$ and $I^+ = I \cap U$. Then

$$I = I^- T_0 I^+ = I^+ T_0 I^-. \tag{3.1}$$

Write $B_0 = B \cap K = T_0 I^+$.

(4) (Iwasawa decomposition) We have

$$G = BK.$$

(5) The group $\mathcal{N}$ is equal to $N_G(T)$, the normalizer of T in G, and every element of the finite Weyl group $W_0 = N_G(T)/T$ has a representative in K.

(6) The set $S_0 = S \cap W_0$ is a set of simple reflections of W_0.

The group G also has the following Bruhat decomposition, which follows from the axioms of a generalized Tits system:

$$G = \sqcup_{w \in W} IwI. \tag{3.2}$$

Lemma 3.3. *(i) There are natural bijections*

$$B_0 \backslash K/I \leftrightarrow I \backslash K/I \leftrightarrow W_0,$$

taking $w \in W_0$ to the double coset $B_0 w I = IwI$.

(ii) We have the decomposition

$$G = \sqcup_{w \in W_0} BwI.$$

Proof. (i) amounts to the Bruhat decomposition for the reduction of K mod π, in light of the compatibility of B and I. Note the importance of the fact that W_0 has representatives in K. (ii) follows from (i) by using the Iwasawa decomposition (so $B \backslash G$ can be identified with $B_0 \backslash K$.) $\square$

4. The Iwahori-Hecke algebra

In this section we explain Bernstein's presentation of the Iwahori-Hecke algebra of G. Write $\mathcal{H}_I$ for $C_c^\infty(I\backslash G/I)$, the space of compactly supported, smooth, and complex-valued functions on $I\backslash G/I$. We fix Haar measure on G so that I has measure 1. Then $\mathcal{H}_I$ is an algebra under convolution multiplication, called the *Iwahori-Hecke algebra* of G. In this section we summarize a presentation of this algebra due to Bernstein, closely following the presentation in [Lu].

The most natural presentation of the Iwahori-Hecke algebra is the one with generators consisting of the characteristic functions of the double cosets IwI, $w \in W$. On the other hand, there is a natural isomorphism

$$W \simeq W_0 \ltimes \Lambda,$$

where W_0 acts on the coweight lattice $\Lambda = X_*(T)$ in the obvious way. We denote this action by $w : \lambda \mapsto w(\lambda)$, for $w \in W_0$, and $\lambda \in \Lambda$. Given an element $\lambda \in \Lambda$, we write e^λ when we regard it as an element of W.

Write $X^*(T)$ for the weight lattice of T, and recall the natural perfect pairing

$$\langle\ ,\ \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}. \tag{4.1}$$

Write $R \subset X_*(T)$ (resp. $\check{R} \subset X^*(T)$) for the set of coroots (resp. for the set of roots), and R^+ (resp. R^-) for the positive (resp. negative) coroots with respect to the Borel subgroup B. Write $\alpha \leftrightarrow \check{\alpha}$ for the natural bijection between coroots and roots. Recall that $\langle \alpha, \check{\alpha}\rangle = 2$.

Write S_0 for the set of simple coroots in R^+. Then $N_G(T)$ acts on T, and hence on $X_*(T)$, and it is well known that for $\alpha \in S_0$ there is an element $s_\alpha \in N_G(T)/T$ whose induced action on $X_*(T)$ is the simple reflection

$$s_\alpha(\lambda) = \lambda \text{-} \langle \lambda, \check{\alpha}\rangle \alpha.$$

We place a partial order on the elements of $\check{R}$ defined by $\check{\alpha}_1 \leq \check{\alpha}_2 \Leftrightarrow \check{\alpha}_2 - \check{\alpha}_1$ is a linear combination of simple roots with non-negative integer coefficients. We write S_m for the set of $\beta \in R$ such that $\check{\beta}$ is a minimal element for $\leq$.

We now define the length function $\ell : W \to \mathbb{N}$. Define for $we^\lambda \in W = W_0 \cdot \Lambda$

$$\ell(we^\lambda) = \sum_{\alpha \in R^+, w(\alpha) \in R^-} |\langle \lambda, \check{\alpha}\rangle + 1| + \sum_{\alpha \in R^+, w(\alpha) \in R^+} |\langle \lambda, \check{\alpha}\rangle|.$$

Write

$$S = \{s_\alpha \mid \alpha \in S_0\} \cup \{s_\alpha e^\alpha \mid \alpha \in S_m\}.$$

Let $\Lambda' \subset X_*(T) = \Lambda$ be the subgroup generated by R. The subgroup $W' = W_0 \cdot \Lambda'$ is a Coxeter group with S as the set of simple reflections, and the length function being the restriction of ℓ. The group Ω (see section 3) is a complement for W' in W, and is equal to the set of elements ρ of W such that $\ell(\rho) = 0$.

We now give Bernstein's presentation of the Iwahori-Hecke algebra; see [Lu] for more details.

Theorem 4.2. *Let H be the algebra with generators*

$$\{T_w, w \in W_0,\ e^\lambda, \lambda \in \Lambda\}$$

subject to the following relations:

(i) $(T_s + 1)(T_s - q) = 0$ for any $s \in S_0$.

(ii) $T_y \cdot T_w = T_{yw}$ if $\ell(y) + \ell(w) = \ell(yw)$, where ℓ is the length function on W_0.

(iii) $e^{\lambda_1} \cdot e^{\lambda_2} = e^{\lambda_1 + \lambda_2}$, for $\lambda_1, \lambda_2 \in \Lambda$.

(iv) For $s = s_\alpha \in S_0$, $\lambda \in \Lambda$ we have

$$T_s e^{s(\lambda)} - e^\lambda T_s = (1 - q) \frac{e^\lambda - e^{s(\lambda)}}{1 - e^{-\alpha}}.$$

Then there is a natural isomorphism

$$\nu : \mathcal{H}_I(I \backslash G / I) \xrightarrow{\sim} H. \tag{4.3}$$

We now describe the isomorphism (4.3).

For $w \in W$, write $\widetilde{T}_w$ for

$$\frac{1}{\mathrm{vol}(IwI)} \,\mathrm{ch}(IwI) \in \mathcal{H}_I,$$

where $\mathrm{ch}(IwI)$ is the characteristic function of IwI. It is well known from the structure theory of $\mathcal{H}_I$ that the functions $\mathrm{ch}(IwI)$ are invertible in $\mathcal{H}_I$.

An element $\lambda \in X_*(T)$ is called *dominant* if $t_\lambda I^- t_\lambda^{-1} \subset I^-$; this is equivalent to saying that $t_\lambda^{-1} I^+ t_\lambda \subset I^+$. If λ_1 and λ_2 are both dominant, then $\widetilde{T}_{\lambda_1} \widetilde{T}_{\lambda_2} = \widetilde{T}_{\lambda_1 + \lambda_2}$. (To simplify notation, we write $\widetilde{T}_{\lambda_i}$ instead of $\widetilde{T}_{t_{\lambda_i}}$.)

Given $\lambda \in \Lambda$, write $\lambda = \lambda_1 - \lambda_2$, with λ_1, λ_2 dominant. Write $\theta_\lambda = \widetilde{T}_{\lambda_1} \widetilde{T}_{\lambda_2}^{-1}$, and note that this is independent of the choice of λ_1 and λ_2.

Then the isomorphism ν in (4.3) is given by the map

$$\widetilde{T}_w \mapsto T_w, \qquad w \in W_0,$$
$$\theta_\lambda \mapsto e^\lambda, \qquad \lambda \in \Lambda.$$

We have the following important result; see [Lu] for a proof.

Proposition 4.4. *The elements $T_w e^\lambda \in H$, $(w \in W_0, \lambda \in \Lambda)$ are linearly independent over $\mathbb{C}$. Consequently the elements $\widetilde{T}_w \theta_\lambda \in \mathcal{H}_I$ are linearly independent over $\mathbb{C}$.*

5. The main result

We maintain notation from the previous section. From now on we shall focus on the simplest component $\Omega_{\mathbf{1}}$ of $\Omega(G)$, given by taking the Levi subgroup M to be the maximal split torus T, and the cuspidal representation of T to be the trivial one-dimensional representation, $\mathbf{1}$. Let $B = TU$, K, and I be as in section 3.

From the discussion in section 2.6 we obtain a representation $\bar{\Pi}$ of T that is a projective generator for the cuspidal component of $\mathcal{S}(T)$ associated to $\mathbf{1}$. Concretely, $\bar{\Pi} = C_c^\infty(T/T_0)$ (the convolution algebra in which T_0 has measure 1), which is isomorphic to the group ring $\mathbb{C}[T/T_0]$. One may regard $\bar{\Pi}$ as the "universal unramified character" of T. It now follows from section 2.9 that the "universal unramified principal series" $\Pi = i_{G,T}(\bar{\Pi})$ is a projective generator for the category $\mathcal{S}(\Omega_{\mathbf{1}})$, where $\Omega_{\mathbf{1}}$ is the component of $\Omega(G)$ corresponding to the cuspidal data $(T, \mathbf{1})$.

Concretely, G acts by right translation on the space of Π, which consists of smooth functions f on G, with values in the ring $\bar{\Pi} = \mathbb{C}[T/T_0]$, that transform under $B = TU$ by

$$f(tug) = \delta(t)^{1/2} t \cdot f(g), \qquad \text{for } t \in T,\ u \in U,\ g \in G. \tag{5.1}$$

Here δ is the modulus character of B; δ is trivial on U and unramified on T. The expression $t \cdot f(g)$ is multiplication in $\bar{\Pi}$, viewing t as an element of $\mathbb{C}[T/T_0]$. If one wishes to view $\bar{\Pi}$ as $C_c^\infty(T/T_0)$, i.e., functions on T, then $\big(f(tug)\big)(t_1) = \delta(t)^{1/2}\big(f(g)\big)(t_1 t)$, for $t_1 \in T$.

By Bernstein's theorem (theorem 2.7), the category $\mathcal{S}(\Omega_{\mathbf{1}})$ is equivalent to the category of modules over $\mathrm{End}_G(\Pi)$. On the other hand, it is known from [Bo] and [Ma] that $\mathcal{S}(\Omega_{\mathbf{1}})$ is also the category of smooth representations of G, all of whose irreducible subquotients have a nonzero vector fixed by I. Moreover, $\mathcal{S}(\Omega_{\mathbf{1}})$ is equivalent to the category of modules over the Iwahori-Hecke algebra $\mathcal{H}_I = C_c^\infty(I\backslash G/I)$. Equivalently, define the "universal representation of G with an I-fixed vector"

$$V_I = c\text{-}ind_I^G(\mathbf{1}) = C_c^\infty(I\backslash G).$$

The following lemma is well known.

Lemma 5.2. *V_I is a projective generator for $\mathcal{S}(\Omega_{\mathbf{1}})$, and moreover*

$$\mathcal{H}_I \xrightarrow{\sim} \mathrm{End}_G(V_I)$$

via left convolution.

Proof. We indicate only a proof of the first statement. Let X be an object of $\mathcal{S}(\Omega_{\mathbf{1}})$. The functor $X \mapsto X^I$ defines the equivalence in [Bo] and [Ma] between $\mathcal{S}(\Omega_{\mathbf{1}})$ and ${}^r\mathfrak{Mod}\,\mathcal{H}_I$. On the other hand, Frobenius reciprocity implies

$$X^I \xrightarrow{\sim} \mathrm{Hom}_G(V_I, X), \tag{5.3}$$

which means that the functor $\mathrm{Hom}_G(V_I, \cdot)$ is exact. Therefore V_I is projective, and as $X \neq 0$ implies $X^I \neq 0$, $\mathrm{Hom}_G(V_I, \cdot)$ is also faithful. It is clear that V_I is finitely generated. □

An immediate consequence of lemma 5.2 is that $\mathcal{H}_I$ and $\mathrm{End}\,_G(\Pi)$ are Morita equivalent. In fact, more is true:

Theorem 5.4. *(i) The projective generators V_I (the universal representation of G with an I-fixed vector) and Π (the universal unramified principal series representation of G) are isomorphic.*

(ii) $\mathcal{H}_I \simeq \mathrm{End}\,_G V_I \simeq \mathrm{End}\,_G \Pi$.

Remark 5.5. The isomorphism in (i) is not entirely new: results essentially equivalent to lemmas 5.9 and 5.10 are to be found in M. Reeder's paper on the unramified principal series [Re] (though the purpose of that paper was quite different), and we are grateful to him for referring us to his paper, and for his suggestions. We also acknowledge the help of A. Roche for some useful points.

The isomorphism in (ii) gives a new construction of the Iwahori-Hecke algebra, which arises from Bernstein's general work on the category of smooth representations of a p-adic group (see [BR]). While the specific techniques for analyzing its structure used below rely on our knowledge of the Iwahori-Hecke algebra and the unramified principal series, its abstract construction is available for every piece of the Bernstein center.

Proof. (ii) follows directly from (i). To prove (i), we construct a G-homomorphism $\phi : V_I \to \Pi$. We then check that ϕ induces an isomorphism of $\mathcal{H}_I$-modules

$$\phi^I : V_I^I \xrightarrow{\sim} \Pi^I.$$

Because ϕ^I is an isomorphism, so is ϕ, due to the equivalence of categories between $\mathcal{S}(\Omega_{\mathbf{1}})$ and ${}^r\mathfrak{Mod}\,\mathcal{H}_I$.

We now define ϕ. Since Π is induced from $\bar{\Pi}$, this amounts to defining a B-equivariant map $\bar{\phi} : V_I \to \bar{\Pi}$. So first define, for $\xi \in V_I = C_c^\infty(I\backslash G)$, an element $\bar{\phi}(\xi) \in \bar{\Pi} = C_c(T/T_0)$, by

$$\bar{\phi}(\xi)(t_1) = \delta(t_1)^{1/2} \int_{u \in U} \xi(t_1 u)\, du.$$

The integral, which converges since ξ is compactly supported, depends only on the coset t_1T_0, since $T_0 \subset I$ and ξ is left-invariant under I. We normalize the measure du so that $I^+ = I \cap U$ has volume 1. Now it is straightforward that

$$\bar{\phi}(tu \cdot \xi) = \delta(t)^{1/2} t \cdot \bar{\phi}(\xi), \tag{5.6}$$

where tu (left-hand side) acts by right translation on ξ, and t (right-hand side) acts by right translation on $\bar{\phi}(\xi)$. Define $\phi(\xi) : G \to \bar{\Pi}$ by

$$\phi(\xi)(g) = \bar{\phi}(g \cdot \xi) = \left(t_1 \mapsto \delta(t_1)^{1/2} \int_{u \in U} \xi(t_1 u g)\, du \right).$$

Using (5.6), one checks directly that $\phi(\xi)$ satisfies (5.1); hence $\phi(\xi) \in \Pi$.

It is clear that ϕ is a G-homomorphism from V_I to Π, and what remains is to show that ϕ is an isomorphism, by looking at its restriction to V_I^I.

The set Π^I consists of functions $f : G \to \bar{\Pi} = \mathbb{C}[T/T_0]$, that transform on the left by B, and that are invariant on the right by I. Such a function is determined by its values on a set of representatives for the double cosets in $B\backslash G/I$. By lemma 3.3 these representatives can be chosen to also give representatives for $B_0\backslash K/I$ and for W_0. Furthermore, we can identify

$$\Pi^I \simeq \{f : B_0\backslash K/I \to \bar{\Pi}\} \quad \text{as complex vector spaces.} \tag{5.7}$$

Indeed, the Iwasawa decomposition and (5.1) imply that $\Pi \simeq \{f : B_0\backslash K \to \bar{\Pi}\}$. Note that the restriction of $f \in \Pi$ to K is actually invariant under B_0.

We shall be more specific about our choice of representatives for W_0, parametrizing the double cosets $B_0\backslash K/I$ and $B\backslash G/I$. Let

$$\rho : W_0 \hookrightarrow K$$

associate to each element of W_0 its chosen representative in K. We can (and will) choose ρ such that

(1) $\rho(1) = 1$,
(2) $\rho(w^{-1}) = \rho(w)^{-1}$.

For $w \in W_0$, we abuse notation and again write w for the corresponding element $\rho(w) \in K$. The reason for being so specific in our choice of ρ is to allow us to make statements such as "$w = 1$," instead of the more laborious "w represents the identity element of W_0."

We are now ready to show that ϕ gives rise to an isomorphism between the $\mathcal{H}_I$-modules Π^I and V_I^I. Let $\xi_0 \in V_I^I$ be the characteristic function of I, and put $f_0 = \phi(\xi_0)$. We obtain, for $w \in W_0$,

$$f_0(w) = \begin{cases} 1 \text{ (i.e., the identity in } \mathbb{C}[T/T_0]) & \text{if } w = 1 \\ 0 & \text{if } w \neq 1. \end{cases} \tag{5.8}$$

To prove the above equality, one verifies that for $t_1 \in T$, $u \in U$ and $w \in W_0$, $t_1 u w \in I$ if and only if $t_1 \in T_0$, $u \in I^+$, and $w = 1$; this uses the decomposition (lemma 3.3) $G = \sqcup BwI$. This is the only calculation that directly involves the definition of ϕ.

We now show that Π^I is a free $\mathcal{H}_I$-module of dimension 1, generated by f_0. On the other hand, it is clear that V_I^I is isomorphic to $\mathcal{H}_{\mathcal{I}}$, and is generated by ξ_0. This will therefore imply that ϕ induces an isomorphism from V_I^I to Π^I, and hence from V_I to Π.

We start by choosing a basis for $\bar{\Pi} = \mathbb{C}[T/T_0]$, consisting of the "monomials" $t_\lambda \in T/T_0$, for $\lambda \in \Lambda$ (see section 3). For $w \in W_0$ and $\lambda \in \Lambda$, write $f_{w,\lambda} : G \to \overline{\Pi}$ for the unique function in Π^I satisfying, for $y \in W_0$,

$$f_{w,\lambda}(y) = \begin{cases} t_\lambda & \text{if } y = w \\ 0 & \text{if } y \neq w \end{cases}$$

(we remind the reader that in this context, W_0 is a set of representatives for $B_0\backslash K/I$). It is clear that the functions $f_{w,\lambda}$ are linearly independent over $\mathbb{C}$, and form a basis for Π viewed as a complex vector space.

Recall that the elements $\widetilde{T}_w\theta_\lambda$, for $w \in W_0$ and $\lambda \in \Lambda$, form a basis for $\mathcal{H}_I$ over $\mathbb{C}$ (proposition 4.4). We now claim that $\widetilde{T}_w\theta_\lambda \cdot f_0$ is a nonzero multiple of $f_{w^{-1},\lambda}$; this will prove that $\Pi^I \simeq \mathcal{H}_I \cdot f_0$. Our claim is an immediate consequence of the following two lemmas:

Lemma 5.9. *If $\lambda_1 \in \Lambda$ is dominant, and $\lambda_2 \in \Lambda$ is arbitrary, then*

$$\theta_{\lambda_1} \cdot f_{1,\lambda_2} = \operatorname{vol}(It_{\lambda_1}I)^{-1}\delta(t_{\lambda_1})^{1/2} f_{1,\lambda_1+\lambda_2}.$$

Since $\theta_\lambda \cdot \theta_{\lambda'} = \theta_{\lambda+\lambda'}$, we conclude that $\theta_{\lambda_1} \cdot f_{1,\lambda_2}$ is a nonzero multiple of $f_{1,\lambda_1+\lambda_2}$ even if λ_1 is arbitrary.

Lemma 5.10. *We have*

$$\widetilde{T}_w \cdot f_{1,\lambda} = \operatorname{vol}(IwI)^{-1} f_{w^{-1},\lambda}.$$

Proof of lemma 5.10: This amounts to the fact that $f \in \Pi$ is determined by its restriction to K. This implies that $\mathcal{H}_I$ acts by the regular representation on functions on $B_0\backslash K/I$, which correspond to vectors in Π^I.

More explicitly, we evaluate $\widetilde{T}_w \cdot f_{1,\lambda}(y)$, for $w, y \in W_0$:

$$\mathrm{vol}(IwI)\widetilde{T}_w \cdot f_{1,\lambda}(y) = \int_{g \in IwI} f_{1,\lambda}(yg)\,dg = \mathrm{vol}(IwI \cap y^{-1}I) \cdot t_\lambda = \begin{cases} t_\lambda & \text{if } y = w^{-1} \\ 0 & \text{if } y \neq w^{-1}. \end{cases}$$

To prove the second equality we must evaluate $f_{1,\lambda}(yg)$. But $yg \in K$, so $f_{1,\lambda}(yg) = t_\lambda$ if and only if $yg \in BI \cap K = I$; otherwise $f_{1,\lambda}(yg) = 0$. The third equality uses the fact that $IwI \cap Iy^{-1}I = \emptyset$ unless $w = y^{-1}$ (and the fact that I has volume 1).

Proof of lemma 5.9: Recall from section 3 that if λ_1 is dominant, then θ_{λ_1} is simply $\mathrm{vol}(It_{\lambda_1}I)^{-1}$ times the characteristic function of $It_{\lambda_1}I$. We first show, for $w \in W_0$, that $\theta_{\lambda_1} \cdot f_{1,\lambda_2}(w) \neq 0$ implies $w = 1$. Indeed, if this value, $\mathrm{vol}(It_{\lambda_1}I)^{-1} \int_{g \in It_{\lambda_1}I} f_{1,\lambda_2}(wg)\,dg$, is nonzero, then $wIt_{\lambda_1}I \cap BI$ must be non-empty (f_{1,λ_1} is supported on BI). If this is so, then we can find $i \in I$ such that $wit_{\lambda_1} \in BI$, whence $wi \in BIt_{\lambda_1}^{-1} \cap K$. Now, by the Iwahori factorization of section 3, $BI = BI^-$, so that

$$BIt_{\lambda_1}^{-1} = BI^-t_{\lambda_1}^{-1} = B(t_{\lambda_1}I^-t_{\lambda_1}^{-1}) \subset BI^- = BI;$$

the inclusion follows from the fact that λ_1 is dominant. But then $wi \in BI$, so $w \in BI$, from which $w = 1$ by lemma 3.3.

Our second step is to evaluate

$$\theta_{\lambda_1} \cdot f_{1,\lambda_2}(1) = \mathrm{vol}(It_{\lambda_1}I)^{-1} \cdot \int_{g \in It_{\lambda_1}I} f_{1,\lambda_2}(g)\,dg.$$

The only elements of G which contribute to this integral are those in $It_{\lambda_1}I \cap BI$. We claim that

$$It_{\lambda_1}I \cap BI = t_{\lambda_1}I. \tag{5.11}$$

Now, f_{1,λ_2} is constant on $t_{\lambda_1}I$, where its value is

$$\delta(t_{\lambda_1})^{1/2}t_{\lambda_1} \cdot f_{1,\lambda_2}(1) = \delta(t_{\lambda_1})^{1/2}t_{\lambda_1} \cdot t_{\lambda_2} = \delta(t_{\lambda_1})^{1/2}t_{\lambda_1+\lambda_2};$$

thus (5.11) implies the assertion of lemma 5.9. Now to verify (5.11), it is enough to prove the inclusion

$$It_{\lambda_1}I \cap BI \subset t_{\lambda_1}I,$$

since the opposite inclusion is obvious. We show more, namely that $It_{\lambda_1}I \cap B \subset t_{\lambda_1}B_0$. Using the Iwahori decomposition, we write

$$I = B_0I^- = I^-B_0 = I^-t_{\lambda_1}^{-1}t_{\lambda_1}B_0.$$

Thus

$$\begin{aligned} It_{\lambda_1}I \cap B &= B_0 I^-(t_{\lambda_1} I^- t_{\lambda_1}^{-1}) t_{\lambda_1} B_0 \cap B \\ &\subset B_0 I^- t_{\lambda_1} B_0 \cap B \quad \text{(since } \lambda_1 \text{ is dominant)} \\ &= B_0 (I^- \cap B) t_{\lambda_1} B_0 \\ &= t_{\lambda_1}(t_{\lambda_1}^{-1} B_0 t_{\lambda_1}) B_0 \quad \text{(since } I^- \cap B = \{1\}\text{; see section 3)} \\ &= t_{\lambda_1} B_0 \quad \text{(since } \lambda_1 \text{ is dominant and } B_0 = T_0 I^+). \end{aligned}$$

This completes the proof of lemma 5.9, and hence the proof of the theorem. □

References

[BeD] Bernstein, J.N.: Le "centre" de Bernstein (rédigé par P. Deligne). Représentations des groupes réductifs sur un corps local. Paris, 1984, pp. 1–32.

[BR] Bernstein, J.N. and Rumelhart, K.: Representations of p-adic groups. Manuscript.

[Bo] Borel, A.: Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Inventiones Math.* 35 (1976), 233–259.

[BK] Bushnell, C., Kutzko, P.: The admissible dual of $GL_n(F)$ via compact open subgroups, Annals of Math. Studies 129, Princeton University Press, 1993.

[Iw] Iwahori, N.: Generalized Tits system (Bruhat decomposition) on p-adic semisimple groups, in *Algebraic Groups and Discontinuous Subgroups.* Proc. Symp. Pure Math. IX. A.M.S. Providence, 1966.

[KL] Kazhdan, D., Lusztig, G.: Proof of the Deligne–Langlands conjecture for Hecke algebras, *Invent. Math.*, 87, (1987) 153–215.

[Lu] Lusztig, G.: Affine Hecke algebras and their graded version, *Jour. Amer. Math. Soc.*, 2, no. 3 (1989), 599–635.

[Ma] Matsumoto, H.: Analyse Harmonique dans les Systèmes de Tits Bornologiques de Type Affine. Springer Lecture Notes #590, Berlin 1977.

[Re] Reeder, M.: On certain Iwahori invariants in the unramified principal series, *Pacific Journal of Mathematics*, 153, no. 2 (1992) 313–342.