

# Nonstandard Proofs of Herglotz, Bochner and Bochner-Minlos Theorems

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## Abstract

*We describe a unified approach to Herglotz, Bochner and Bochner-Minlos theorems using a combination of Daniell integral and nonstandard analysis. The proofs suggest a natural extension of the last two theorems to the case when the characteristic function is not continuous. This extension is proven and is demonstrated to be the best one possible.*

The goal of this paper is to show how the classic theorems of Herglotz, Bochner and Bochner-Minlos are in fact simple consequences of the trivial discrete case, i.e. the obvious analogue of these theorems for functions on  $\mathbb{Z}_n$ . It turns out that Herglotz follows immediately (Theorem 1), while Bochner and Bochner-Minlos require a little more work for topological reasons (Theorems 2 and 4). We shall use a combination of Daniell integral and nonstandard analysis to demonstrate the claims above<sup>1</sup>. We shall also show that Bochner and Bochner-Minlos theorems can be extended to the case when the characteristic function is not continuous (in the Herglotz case, since the domain has discrete topology, no function is discontinuous), and shall prove that this extension is the best one can hope for (Theorem 3).

Let us start with the simplest case, the discrete one:

**Lemma:** Let  $\{\hat{\mu}(k)\}_{k=-N'}^{N'}$  be a positive-definite collection of complex numbers<sup>2</sup>, such that  $\hat{\mu}(0) = 1$ , then there is a positive collection of real numbers  $\{\mu(z)\}_{z=-N'}^{N'}$  such that

$$\hat{\mu}(k) = \frac{1}{N'} \sum_{z=-N'}^{N'} e^{-2\pi i \frac{kz}{N'}} \mu(z), \quad (1)$$

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<sup>1</sup>The reader who is unfamiliar with the Daniell approach to integration should consult e.g. chapter 13 of [1]. We will only use rather elementary facts from nonstandard analysis, to which the first half of [2] is a thorough yet leisurely introduction. Alternatively, the first chapter of [3] contains all the material needed in a condensed form.

<sup>2</sup>Positive-definiteness is meant in the sense of positive-definite functions (of  $k$  in this case) [4] and not in the sense of having positive values.

where  $N = 2N' + 1$ .

**proof:** Let

$$\mu(z) = \sum_{k=-N'}^{N'} e^{2\pi i \frac{kz}{N}} \widehat{\mu}(k).$$

If we think of  $e^{2\pi i \frac{kz}{N}}$  as being a vector with  $N$  components, the components being labelled by  $z$ , then it is easy to see, by using the Vandermonde identity, that this set of  $N$  vectors (labeled by  $k$ ) is a basis for  $\mathbb{C}^N$ . Thus, for  $a \in [-N', N']$ , letting  $\delta_a(z)$  be the function which is equal to 0 everywhere except at  $a$ , where it is equal to one, we know, that we can solve the equation

$$\delta_a(z) = \sum_{k=-N'}^{N'} e^{2\pi i \frac{kz}{N}} \widehat{\delta}_a(k)$$

and obtain the complex coefficients  $\widehat{\delta}_a(k)$ .

Consequently, we have

$$\begin{aligned} \mu(a) &= \sum_{z=-N'}^{N'} \mu(z) \delta_a(z) \\ &= \sum_{z=-N'}^{N'} \mu(z) |\delta_a(z)|^2 \\ &= \sum_{z=-N'}^{N'} \sum_{k_1=-N'}^{N'} \sum_{k_2=-N'}^{N'} \mu(z) \overline{\widehat{\delta}_a(k_2)} \widehat{\delta}_a(k_1) e^{2\pi i \frac{(k_1-k_2)z}{N}} \\ &= N \sum_{k_1=-N'}^{N'} \sum_{k_2=-N'}^{N'} \widehat{\mu}(k_2 - k_1) \overline{\widehat{\delta}_a(k_2)} \widehat{\delta}_a(k_1) \\ &\geq 0 \end{aligned}$$

where we used the positive-definiteness of  $\widehat{\mu}(k)$  in the last line. It follows that  $\mu(z) \geq 0$  for all  $z \in [-N', N']$ .

Finally, the fact that  $\mu(z)$  satisfies (1) follows at once from the formula (which was just used above)

$$\frac{1}{N} \sum_{z=-N'}^{N'} e^{\frac{2\pi iz(k-k')}{N}} = \delta_{kk'}. \quad (2)$$

This concludes the proof of the lemma  $\square$

Having dealt with the trivial, discrete case, let us see how the various continuum versions follow from it. Let us begin with the simplest case which is

**Theorem 1 (Herglotz):** Let  $\{\widehat{\mu}(k)\}_{k=-\infty}^{\infty}$  be a positive-definite sequence of complex numbers, such that  $\widehat{\mu}(0) = 1$ , then there is a Borel probability measure  $\mu$  on  $S^1$  such that  $\widehat{\mu}(k) = \int_{-\pi}^{\pi} e^{-ikx} d\mu(x)$ .

**proof:** Let  $N$  be an odd unlimited hyper-natural. Since it is odd there is a hyper-natural  $N'$  such that  $N = 2N' + 1$ . Any real-valued sequence admits a canonical extension to all hyper-integers, which, as is customary, we shall also denote by  $\widehat{\mu}(k)$ . Now, for any hyper-integer  $z \in [-N', N']$ , let

$$\mu(z) = \sum_{k=-N'}^{N'} e^{2\pi i \frac{kz}{N}} \widehat{\mu}(k).$$

Then, using the transfer principle, it follows from the previous lemma that  $\mu(z)$  is positive and that it satisfies (1). Note that this in particular implies that  $1 = \widehat{\mu}(0) = \frac{1}{N} \sum_z \mu(z)$

We want to use  $\mu(z)$  to obtain a measure on the circle, thus we rescale the variable of  $\mu$ . Let  $x = \frac{2\pi z}{N}$ . Note that the  $x$ 's are a lattice of points stretching from infinitesimally above  $-\pi$  to infinitesimally below  $\pi$  with an infinitesimal step, and thus, can be naturally thought of as a lattice of points on  $S^1$ . To reduce clutter we shall write  $\mu(x)$  for  $\mu(z(x))$ . We are now ready to define our Daniell integral.

For our vector algebra we take the vector algebra of continuous real-valued functions on  $S^1$ . We let our integral  $I$  act on an element  $f$  of this algebra via

$$I(f) = \text{st} \left( \frac{1}{N} \sum_x f(x) \mu(x) \right) \quad (3)$$

Here  $f(x)$  is the value of the hyper-real extension of  $f$  at the lattice point  $x$  and  $\text{st}$  stands for the standard part map. It is legitimate to take the standard part since the expression in brackets is a limited hyper-real due to the fact that  $f$  is bounded on  $S^1$  and that  $\frac{1}{N} \sum_x \mu(x) = 1$ .

It follows at once that the above expression is linear and positive, moreover  $I(1) = 1$ . We will be done<sup>3</sup> if we show that any monotone decreasing sequence of elements  $f_n$  in our vector algebra which converges pointwise to

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<sup>3</sup>See page 287 of [1].

0 satisfies  $I(f_n) \rightarrow 0$ . This however is an immediate consequence of the fact that any such sequence converges uniformly (Dini's theorem)<sup>4</sup>. Thus  $I$  is indeed an integral with respect to a Borel probability measure on  $S^1$ .

Finally, the fact that  $\int e^{-ikx} d\mu(x) = \hat{\mu}(k)$  follows at once from the observation that both  $\cos(kx)$  and  $\sin(kx)$  belong to the vector algebra and thus to calculate the integral of  $e^{-ikx} = \cos(kx) - i \sin(kx)$  one can use (3), which combined with (1) and (2) leads to the desired conclusion, and the proof is complete  $\square$

The interested reader should compare the proof above to the one given in [5].

How was nonstandard analysis utilized in the proof? On one hand we used it to conclude positivity of our tentative measure as well as have a Fourier inversion formula for it since these follow immediately by transfer from the discrete case.

On the other hand, we used a nonstandard expression in order to construct the Daniell integral. The formula (3) is just a weighted average of the values of the function at a lattice of points, an expression familiar to anyone dealing with trigonometric sums. Typically, one would like to make the lattice denser and denser by taking the number of points to infinity. The key advantage that nonstandard analysis furnishes is that one does not need to worry about the existence of a limit as  $N$  is 'taken to' infinity since we can just set it to an unlimited value. The reader should note the important subtle point that the integral above is a standard one as it acts on standard functions. The nonstandard analysis is only a device to define this action.

Incidentally, it follows from the proof above that in fact any positive Borel measure has a representation as a hyper finite sum [6] when integrated against a continuous function. To demonstrate this note that the Fourier coefficients of any such measure are a positive-definite sequence. Assume that  $f$  is a smooth function, given in terms of its Fourier expansion  $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{-ikx}$  (smoothness is only needed to guarantee uniform convergence of the series). Note that this means that  $f(x) \approx \sum_{k=-N'}^{N'} \hat{f}(k)e^{-ikx}$ , where  $N'$  is the unlimited hyper-integer from the proof above. We have

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<sup>4</sup>Note that this phrase is nothing but a demonstration of the Riesz-Markov theorem! Its proof becomes remarkably simple in the Daniell's approach to integration.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) d\mu(x) &= \sum_{k=-\infty}^{\infty} \hat{f}(k) \int_{-\pi}^{\pi} e^{-ikx} d\mu(x) \\
&= \sum_{k=-\infty}^{\infty} \hat{f}(k) \hat{\mu}(k) \\
&= \text{st} \left( \sum_{k=-N'}^{N'} \hat{f}(k) \hat{\mu}(k) \right) \\
&= \text{st} \left( \sum_{k=-N'}^{N'} \sum_x \hat{f}(k) e^{-ikx} \frac{\mu(x)}{N} \right) \\
&= \text{st} \left( \frac{1}{N} \sum_x f(x) \mu(x) \right).
\end{aligned}$$

Therefore, the integral of any smooth function is indeed given by (3). Since the smooth functions are a dense subset in the space of all continuous functions (in the uniform topology), the above formula holds for continuous  $f$  as well. Thus, the integral of any continuous function with respect to any positive Borel measure on  $S^1$  is indeed given by a nonstandard weighted sum.

Finally, the reader familiar with the subject should note that the Daniell integral above could have been replaced with a Loeb measure construction [2, 3, 7, 8]. So, instead of defining the Daniell integral, we would define the Loeb measure as a weighted counting one and then push it down with the standard part map. This also applies to the theorems that follow. The reason for not doing so was the desire to keep the nonstandard parts of the proofs as elementary as possible.

Let us move on now to Bochner's theorem. We shall prove in fact an extension of it to the case of a discontinuous characteristic function. The original theorem of Bochner is part (b) of the following

**Theorem 2:** Let  $\hat{\mu}(q)$  be a positive-definite real-valued function of a *rational* variable such that  $\hat{\mu}(0) = 1$ , then

- a- There is a Borel probability measure  $\mu$  on  $\beta\mathbb{R}$  (the Stone-Ćech compactification of  $\mathbb{R}$ )<sup>5</sup> such that  $\hat{\mu}(q) = \int_{\beta\mathbb{R}} e^{-iqx} d\mu(x)$ . Here  $e^{-iqx}$  stands for the canonical extension of the bounded continuous function  $e^{-iqx}$  to the compactification.

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<sup>5</sup>For a standard reference on  $\beta\mathbb{R}$  see for example page 237 of [9]. The construction of  $\beta\mathbb{R}$  through bounded continuous functions described there will be utilized in the proof of Theorem 3 below.

- b- (Bochner)<sup>6</sup> If additionally the function  $\widehat{\mu}(q)$  is continuous at 0 (in the induced topology), then in fact the corona set  $(\beta\mathbb{R} - \mathbb{R})$  has measure 0 and thus the measure  $\mu$  can be naturally considered to be a measure on  $\mathbb{R}$ .

**proof:**

- a- As before, choose an odd, unlimited hyper-natural  $N = 2N' + 1$ , however, this time let  $N' = (N''!)^2$  (the reason for this strange choice will be apparent momentarily). Now, let  $k$  and  $z$  be both hyper-integers in  $\{-N', \dots, N'\}$ . Note that as  $k$  varies on the specified range,  $\frac{k}{N''!}$  goes over a subset of hyper-rationals which includes all standard ones. Extend the given function  $\widehat{\mu}(q)$  to all hyper-rationals in the canonical way. As before we shall write  $\widehat{\mu}(k)$  for  $\widehat{\mu}(q(k))$  where  $q(k) = \frac{k}{N''!}$ . Proceeding as in the proof of Herglotz's theorem, we obtain  $\mu(z)$  and find that it is positive for all  $z \in [-N', N']$ . However, instead of the relation used previously, let  $x = \frac{2\pi N''!z}{N}$ . In other words we now scale the variable so that it spans the whole real line (and more, as  $x$  will take unlimited values as well) with an infinitesimal step. Again, we will write  $\mu(x)$  for  $\mu(z(x))$ . We are now ready to define our Daniell integral.

Take as your vector algebra the set of all continuous functions on  $\beta\mathbb{R}$ . Define the integral of an element of this algebra via formula (3) as before. Note, that since  $\beta\mathbb{R}$  is compact, Dini's theorem applies again and we obtain our measure. Finally, the fact that its Fourier transform is exactly  $\widehat{\mu}(q)$  is proven the same way as before (this is why we needed  $\frac{k}{N''!}$  to go over all rationals).

- b- If we know that  $\widehat{\mu}(q)$  is continuous at 0 then we know that its hyper-real extension is infinitesimally close to  $\widehat{\mu}(0) = 1$  for infinitesimal  $q$ 's. In other words, let  $A$  be any hyper-natural in  $[1, N']$  such that  $\frac{A}{N''!}$  is infinitesimal, then  $\widehat{\mu}(A)$  is infinitesimally close to 1. But,

$$\widehat{\mu}(A) = \frac{1}{N} \sum_{z=-N'}^{N'} \mu(z) e^{2\pi i \frac{Az}{N}}. \quad (4)$$

This is of course nothing but (1), for  $k = A$ . Note that  $\widehat{\mu}$  is written here in terms of  $k$  and not  $q$ .

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<sup>6</sup>The formulation given here seems to be different from the usual one. However, the two formulations are in fact equivalent as is easy to show using the property that positive-definite functions are uniformly continuous if continuous at 0 (see page 18 in [4]).

For this sum to be infinitesimally close to one it follows at once that the sum of the coefficients of the phases lying in the left half-plane must be infinitesimal. It follows that,

$$\frac{1}{N} \sum_{z: \frac{|z|^A}{N} \geq \frac{1}{4}} \mu(z) \approx 0$$

Recalling the relation between  $x$  and  $z$  we see that it follows that

$$\frac{1}{N} \sum_{x: |x| \geq \frac{\pi}{2} \frac{N''!}{A}} \mu(x) \approx 0$$

Since  $A$  was arbitrary as long as  $\frac{A}{N''!}$  is infinitesimal we have that for any unlimited  $B$ ,  $\frac{1}{N} \sum_{x: |x| \geq B} \mu(x) \approx 0$ .

Thus, by underflow, for any real  $\alpha \in (0, 1)$ , there is a limited natural  $C$  such that  $\sum_{|x| \leq C} \mu(x) > \alpha$ . If we take now a positive, continuous function  $f$  which is equal to 1 on  $[-C, C]$ , 0 on the complement of  $[-C-1, C+1]$  and is less than or equal to 1 everywhere, we see that  $I(f) \geq \alpha$ , from which it follows that the measure of  $[-C-1, C+1] \geq \alpha$ . Since  $\alpha$  is arbitrary we have that the measure of  $\mathbb{R} = 1 \square$

It is natural to wonder at this moment whether part (a) of the theorem above can be improved upon. Thus, one may ask : Is there a 1-1 correspondence between probability measures on  $\beta\mathbb{R}$  and positive definite functions of rationals? If not, then, is the situation improved if one takes positive definite functions of reals? Alas, the answer is no in both cases and it turns out that the result above is essentially the best one can have, in view of the following somewhat surprising

**Theorem 3:** The map taking a probability measure on  $\beta\mathbb{R}$  to its Fourier transform is neither surjective onto nor injective into positive-definite functions of reals.

**proof:** Let us start with surjectivity. We shall show that the positive-definite  $\hat{\mu}$  which is equal to 1 on rationals and 0 on irrationals is not the Fourier transform of any probability measure on  $\beta\mathbb{R}$ . Assume it is. Then, since the measure of  $\beta\mathbb{R}$  is equal to 1, it follows that the support of  $\mu$  is contained in the set  $A$ , where  $A = \{x \in \beta\mathbb{R} : e^{ikx} = 1, \forall k \in \mathbb{Q}\}$ . Clearly, there is only one such point in  $\mathbb{R}$  which is  $k = 0$ . Does  $A$  contain any points from the corona set? It turns out that it doesn't.

To see this, assume it does, and let this point be  $x_\alpha$  where the index ranges over continuous bounded functions on  $\mathbb{R}$  (we are using here the construction of the Stone-Ćech compactification in terms of bounded continuous functions and Tychonoff topology). We know that all the components corresponding to  $e^{ikx}$  for rational  $k$  must be equal to 1, what are the values of the other components? Choose some bounded, continuous function  $f$ . We shall find what the value of the component corresponding to  $f$ , denoted by  $x_f$ , is. Since  $x_\alpha$  is supposed to be in the corona set, it means any neighbourhood of it intersects  $\mathbb{R}$ . In other words, for any  $\epsilon > 0$  and any integer  $k$  there should be a point  $x$  in  $\mathbb{R}$  such that both  $e^{ikx}$  and  $f(x)$  are within  $\epsilon$  of 1 and  $f(x_f)$  respectively. If we let  $f$  be any monotone bounded function, it follows that  $x_f = 0$ . However, an immediate consequence of this is that for any *other* bounded, continuous function  $g$ ,  $x_g = 0$  as the fact that  $x_f = 0$  constrains the allowed values of  $x_g$  to an arbitrarily small neighbourhood of 0. Therefore, for any function, the value of the corresponding component of  $x_\alpha$  is equal to the value of the function at 0, which of course means that the point is equal to 0 (under the usual imbedding of  $\mathbb{R}$  into  $\beta\mathbb{R}$ ).

We have thus seen that our measure  $\mu$  is just the Dirac measure at 0, and thus has a Fourier transform equal to 1 everywhere. A contradiction.

Let us now consider injectivity. It is easy to see that  $\mathcal{A} = \{e^{ikx} : k \in \mathbb{R}\}$  vanishes nowhere on  $\beta\mathbb{R}$ . If it separated points, it would follow by Stone-Weierstrass that finite linear combinations of elements of  $\mathcal{A}$  are uniformly dense in the space of all continuous functions on  $\beta\mathbb{R}$ . This, would imply that finite linear combinations of elements of  $\mathcal{A}$  are uniformly dense in the space of all bounded continuous functions on  $\mathbb{R}$ . This however cannot be the case in view of the Poincare recurrence theorem on the  $n$ -torus. This is because, if you take any value of any finite linear combination of elements of  $\mathcal{A}$ , then that value is re-approached arbitrarily closely if one adds sufficiently many multiples of  $2\pi$  to the variable. Therefore, it is impossible to uniformly approximate a gaussian for example. Thus  $\mathcal{A}$  does not separate points, i.e. there are  $x_1, x_2 \in \beta\mathbb{R}$  such that  $\forall k \in \mathbb{R} : e^{ikx_1} = e^{ikx_2}$ . Taking now our measures to be Dirac ones supported at  $x_1$  and  $x_2$ , we see that they both have the same Fourier transform  $\square$

As a final remark, note that the proofs of Theorems (2) and (3) above carry over easily (essentially verbatim) to functions of  $\mathbb{Q}^n$  and measures on  $\beta(\mathbb{R}^n)$  or  $\mathbb{R}^n$  respectively.

Finally, let us discuss the Bochner-Minlos theorem. We shall prove it only for the space of distributions on  $\mathbb{R}$ . However the same proof carries over to any nuclear space with a suitable basis and inner product. So, consider

**Theorem 4 (Bochner-Minlos):** Let  $\widehat{\mu}(J)$  be a positive-definite function on  $\mathcal{S}$  (the Schwartz space on the real line) which is continuous at 0, then there is a probability measure  $\mu$  on the space of tempered distributions  $\mathcal{S}'$ , such that its Fourier transform is  $\widehat{\mu}(J)$ .<sup>7</sup>

**proof:** We will use the well-known fact that  $\mathcal{S}$  is isomorphic to the space of all sequences of fast decrease (see for example page 143 in [10]). Thus  $\widehat{\mu}$  can be considered as a real valued function of these sequences. Restrict now this function to the set of finite sequences (which are obviously of fast decrease) of rationals. Again, this function (of finite sequences of rationals) extends in a canonical way to a function of all hyper-finite sequences of hyper-rationals. Fix some unlimited hyper-natural  $A$ . We thus obtain from  $\widehat{\mu}(J)$  a function from the set of all hyper-finite sequences  $\{J_n\}_{n=1}^A$ , of hyper-rationals of unlimited length  $A$ . This function will also be denoted by  $\widehat{\mu}$ .

Now fix another unlimited hyper-natural  $B$  and for any hyper-natural  $n \in [1, A]$  we let  $N_n = 2N'_n + 1$ , where  $N'_n = (N''_n!)^2$  and  $N''_n = B!n^{\frac{B(B+1)}{2}}$ . As before, we let  $k_n$  and  $z_n$  be hyper-integers in  $[-N'_n, N'_n]$ . Finally, let the relation between  $J_n$  and  $k_n$  be  $J_n = \frac{k_n}{N''_n!}$  and between  $x_n$  and  $z_n$  be  $x_n = \frac{2\pi N''_n! z_n}{N_n}$ . We shall denote all of the above hyper-finite sequences by the corresponding bold letters. The reason for the relation between  $x_n$  and  $z_n$  is the same as it was for the relation between  $x$  and  $z$  in theorem 2 (see the proof of part (a) of Theorem 2 above), while the relation between  $J_n$  and  $k_n$  is the same as the relation between  $q$  and  $k$  there. The new ingredient here is that  $N''$  is dependent on  $n$ . The rationale for the particular form of this dependence will be apparent below.

Now, we can proceed as before and starting from  $\widehat{\mu}(\mathbf{k})$  obtain  $\mu(\mathbf{x})$  as well as deduce that the latter is always positive, the only difference is that the sums now go over a grid with  $A$  axes instead of a one dimensional lattice. We define our Daniell integral exactly as before: our vector algebra is the set of bounded continuous functions on  $\mathcal{S}'$  and our integral is given by (3). It is natural to try to proceed as in the proof of Bochner's Theorem above and to use (3) to define an integral on the Stone-Ćech compactification first and then to show that the corona set has measure zero. However, this procedure doesn't quite work now since  $\mathcal{S}'$  is not locally-compact and thus is not an open subset of  $\beta\mathcal{S}'$ . We need to maneuver a little differently in this more complicated case. Thus, let us first prove an analogous property of the  $\mu(\mathbf{x})$  as in the Bochner case above.

Let  $C_k$  be the set of all real-valued sequences  $a_n$  such that  $|a_n| \leq kn^k$ . We

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<sup>7</sup>We are only proving here the analogue of (b) of Theorem 2. The analogue of (a) as well as Theorem 3 have essentially identical proofs.

shall also denote by the same symbol the set of all hyper-real, hyper-finite sequences of length  $A$  satisfying the same property (it shall be clear from the context which one is being used). Note, that  $\mathcal{S}' = \bigcup_{k=1}^{\infty} C_k$  and that each  $C_k$  is actually compact in the standard topology on  $\mathcal{S}'$  (see page 166 in [10]). We claim that for any real  $\alpha \in (0, 1)$  there is a standard  $k$  such that  $\sum_{\mathbf{x} \in C_k} \mu(\mathbf{x}) > \alpha$ . The proof of this fact is identical to the one given in theorem 2 if one notes that  $\widehat{\mu}(\mathbf{J})$  should be infinitesimally close to 1 if  $J_n = \frac{1}{kn^k}$  for any unlimited  $k \leq B$ . Before we prove that this is the case for such  $\mathbf{J}$ 's note that the need to ensure that such  $\mathbf{J}$ 's are in the domain of  $\widehat{\mu}$  is what dictated the choice of the peculiar form of  $N_n''$  above. Observe that<sup>8</sup>  $N_n'' = (1n^1)(2n^2)(3n^3) \dots (Bn^B)$  and thus indeed for any  $k \leq B$  the denominator of  $J_n$  has the needed factor  $kn^k$  while the numerator  $k_n$  can be chosen to cancel all the other factors.

The quickest way to see that  $\widehat{\mu}(\mathbf{J})$  of such  $\mathbf{J}$ 's is infinitesimally close to 1 is to note that  $\widehat{\mu}$  is continuous and thus if  $\mathbf{J}$  is infinitesimally 'close' to 0, its image under  $\widehat{\mu}$  is infinitesimally close to  $\widehat{\mu}(0) = 1$ . 'Infinitesimally close' here means that the distance from this  $\mathbf{J}$  to 0 is infinitesimal, where the distance is given by the canonical extension of the usual metric on a Fréchet space (see pages 131 and 143 in [10])<sup>9</sup>. Since the first finitely many semi-norms entering the definition of the metric can be made arbitrarily small by taking  $k$  sufficiently large, the result follows at once.

We are now ready to prove that (3) does indeed define a Daniell integral. To this end take a sequence of bounded continuous functions  $f_n$  on  $\mathcal{S}'$  decreasing monotonically to 0 and choose a real  $\epsilon > 0$ . Choose a  $k$  such that  $\sum_{\mathbf{x} \notin C_k} \mu(\mathbf{x}) < \frac{\epsilon}{2M}$  where  $M$  is an upper bound for  $f_1$ . Now, applying Dini on the compact  $C_k$  we see that there is an  $l$  such that  $\forall n > l$  we have that  $f_n(\mathbf{x}) < \frac{\epsilon}{2}$  for all  $\mathbf{x} \in C_k$ . Combining the above we have that  $\forall n > l$

$$\begin{aligned} \sum_{\mathbf{x}} f_n(\mathbf{x})\mu(\mathbf{x}) &= \sum_{\mathbf{x} \in C_k} f_n(\mathbf{x})\mu(\mathbf{x}) + \sum_{\mathbf{x} \notin C_k} f_n(\mathbf{x})\mu(\mathbf{x}) \\ &< \sum_{\mathbf{x} \in C_k} \frac{\epsilon}{2}\mu(\mathbf{x}) + \sum_{\mathbf{x} \notin C_k} M\mu(\mathbf{x}) \\ &< \frac{\epsilon}{2} \sum_{\mathbf{x}} \mu(\mathbf{x}) + M \sum_{\mathbf{x} \notin C_k} \mu(\mathbf{x}) \\ &< \epsilon. \end{aligned}$$

Thus the sequence of integrals of  $f_n$ 's converges to zero and we have our measure. It follows immediately that the Fourier transform of this measure

<sup>8</sup>This is hyper-finite multiplication not a limit.

<sup>9</sup>The proof of this fact is essentially identical with that of the nonstandard criterion of continuity for a function of one variable.

coincides with the given function  $\widehat{\mu}$  at the finite sequences of rationals, and since these are dense in  $\mathcal{S}$ , and  $\widehat{\mu}$  is continuous, this equality extends to the whole space  $\square$

The reader should appreciate the similarity between the proofs above. The argument would in general have three steps:

- Usage of positive definiteness combined with the lemma to show positivity of the would-be measure.
- Application of ‘Fourier inversion’ (the formula defining  $\mu(z)$ ) to construct the measure which is then used to define the Daniell integral (formula (3)). It should be emphasized once again that the integral is a standard one with nonstandard analysis only being utilized to define it.
- The integral thus defined is showed to satisfy the axioms of Daniell integration.

The only essential difference between the theorems above is contained in the last step. This difference is in turn a consequence of the difference of the topological properties of the spaces under consideration. Herglotz deals with the compact case, Bochner with the non-compact one which however is still locally compact, and finally Bochner-Minlos handles the non-locally-compact situation.

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