Learning Development

University of Northampton

RATES OF CHANGE

DISTANCE, TIME, SPEED AND SUCH

Speed is a rate of change. It is a rate of change of distance with time and can be measured in miles per hour (mph), kilometres per hour (km/h), meters per second (m/s) and many other ways.

A hot body cools. The rate of cooling can be measured in:

degrees Celsius per second (^{0}C/s).

A violin string vibrates. The rate of vibration can be measured in cycles per second (c/s),;

but this unit is given the special name Hertz (Hz).

All the changes so far are measured against time, and this is the standard we are most likely to use. But there are plenty of the others.

A road changes in height. The gradient of a hill is measured in height gained against distance travelled: 1 in 10 means 1 meter gained in height for 10 meters travelled. This could be expressed as 0.1 meters per meter; but it makes no difference whether we're talking about metres, feet, yards, centimetres or inches, so just 0.1 will do, or 1 in 10,
or (as the road signs say nowadays) 10%.

The height could be measured using an altimeter. They usually depend on the variation in pressure with height. What units would that rate of change be measured in?

Exercise 1

Make up your own, for example relating to pressure – there are lots of other possibilities.

Now check your answers.

As we look around, the intensity of the light we see varies. So we could have a rate of change of light intensity per degree, measured perhaps in lumens/ degree.

SUMMARY

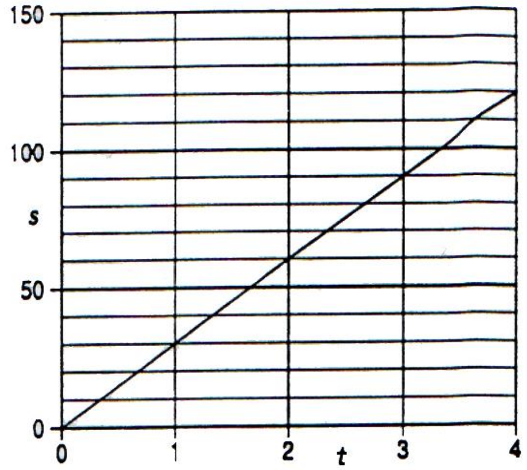
Calculus is first of all about rates of change.

Examples of rates of change are:

- Speed – rate of change of distance with time.
- Cooling and heating – rate of change of temperature with time.
- Gradient of a road – rate of change of height with distance.
- Air pressure 'gradient' – rate of change of pressure with height.

Straight line graphs

Here is a graph of distance from the Watford Gap Service station, plotted against time as a car travels South on the M1. Distance (symbol *s*) is in miles and time (symbol *t*) is in hours.



Exercise 2

From your knowledge of graphs, write down the equation of the straight line. Don't forget to use *s* and *t* instead of y and x.

Now check your answer.

Exercise 3

Now write down the gradient of the straight line.

Now check your answer.

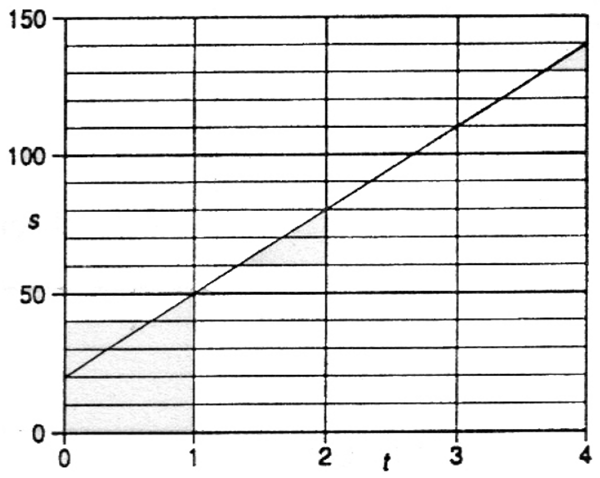
Gradient is $\dfrac{\text{change [illegible]}}{\text{change [illegible]}}$

So in this case the **rate of change** of distance with time (the velocity) is 30mph. And because the connection between distance and time is expressed as a graph, you can call this rate of change the **gradient** of the graph. (It is rather unrealistic to give a 30mph as a motorway average speed, but perhaps one day the roadworks will clear away and it will be possible to achieve or even exceed this.)

Remember that this gradient is not quite the same as the gradient of a road. Gradient of a road is gain in height (change in the vertical variable) divided by distance up to the slope. In this pack, from now on, gradient will always mean vertical change divided by horizontal change.

Exercise 4

Here is another graph of distance (miles) against time (hours).



What is the equation of the line?

Now check your answers.

Exercise 5

What is the gradient of the line?

Now check your answer.

The point of that was to emphasise that you don't change the gradient of a line just by moving the line up and down vertically. It also reminds you that the gradient is not $\frac{s}{t}$ (or $\frac{y}{x}$). It happens that in the first graph $\frac{s}{t}$ would have given the right answer; but in the second case $\frac{s}{t}$ does **not** give the right answer.

$$\frac{y}{x} \quad \text{is not the same as} \quad \frac{\text{change in } y}{\text{change in } x}$$

We will now use *y* and *x* as the variable names. These are the names usually used in maths. Away from pure maths, in the practical world, all sorts of symbols are used; so it's important not to get *y* and *x* too fixed in the mind. But when we're dealing with maths which can be applied to a huge range of subjects, it's better to use symbols which have no special significance.

Here is another equation: $y = 7x + 5$

The gradient of this line is 7.

Exercise 6

State the gradient of the following lines:

a) y = 7 x – 2

b) y = 7 x

c) y = -2 x + 5

d) y = 0.3 x – 2.4

e) y = $\frac{2x}{3}$ - 1

f) y = 5 – 0.24v

Now check your answers.

HELLO d y d x

The symbol $\frac{dy}{dx}$ represents the gradient.

$\frac{dy}{dx}$ can be looked on as shorthand for change in y/ change in x, now here are some 'formulas' for y, followed by the corresponding formulas for $\frac{dy}{dx}$:

$y = 3x + 5$	$\frac{dy}{dx} = 3$
$y = -\frac{2x}{7} + 1$	$\frac{dy}{dx} = -\frac{2}{7}$
$y = 1.6x - 2.9$	$\frac{dy}{dx} = 1.6$

Exercise 7

Finish off the following by writing the formula for $\frac{dy}{dx}$:

a) $y = 2x - 9$

b) $y = 5.3x + 3.87$

c) $y = -5x - 4$

d) $y = 6x$

e) $y = \frac{3x}{8} + 2$

f) $y = ax + b$
where a and b are constants.

Now check your answers.

So in general, *if* $y = mx + c$, then $\frac{dy}{dx} = m$.

Or, if we used *a* and *b* instead of *m* and *c*, if $y = ax + b$ then $\frac{dy}{dx} = a$.

CONSTANTS AND VARIABLES

At this point some of you may perhaps begin to worry as letters start to multiply and hard facts and figures appear to be fading. You would be quite right to worry. When looking at a mixed assortment of letters it is very easy to lose touch with what's going on. It's easy, for example, to forget which letters represent **variables** and which letters represent **constants**.

In the equations, ***x*** and ***y*** are variables. They vary, they are meant to vary. In the equations as written, ***x*** varies and ***y*** follows. The equations make it easy for you to put in values of *x* and calculate values of y – it would be a bit harder to find *x*, given y. therefore ***x*** is sometimes referred to as the 'independent variable' and ***y*** as the 'dependent variable'.

Most of the equations have numbers in, which are known as the 'constants'. In the last equation ***a*** and ***b*** stand for constants. You can think of ***a*** as 'a number' and ***b*** as 'another number'. So ***a*** might be 5 and ***b*** might be 2.3.

It is common practice to use letters from the second half of the alphabet to represent variables*: x, y, v, t* and so on. Letters from the first half of the alphabet are used for constants. (Remember the quadratic equation $ax^2 + bx + c = 0$.)

This is not a strict rule, but is common practice, and should help you when you meet a mixed bag of letters. There is a common exception, and that is *n* when used as a power, as in x^n. In this case *n* is a constant, even though it comes from the second half of the alphabet (just).

Exercise 8

In the following equations, which letters do you think are the constants and which are the variables (in the absence of further information you can only guess)?

a) $y = ax^2 + bx + c$

b) $v = gt - at^3$

c) $x^2 + y^2 = a^2$

d) $y = mx + c$

e) $r = kt^n$

f) $z = \frac{ax}{(x^2 + y^2)}$

Now check your answers.

SUMMARY

The gradient of a straight line is $\frac{\text{change in vertical}}{\text{change in horizontal}}$

In the case of an x – y graph this is $\frac{\text{change in } y}{\text{change in } x}$

The gradient of a distance – time graph gives the speed.

The gradient of an x – y graph is known as $\frac{dy}{dx}$ (dee-wye dee-exe).

The equation of a straight line graph is often written as $y = mx + c$.
m and c are constants; m is the gradient; x and y are variables.

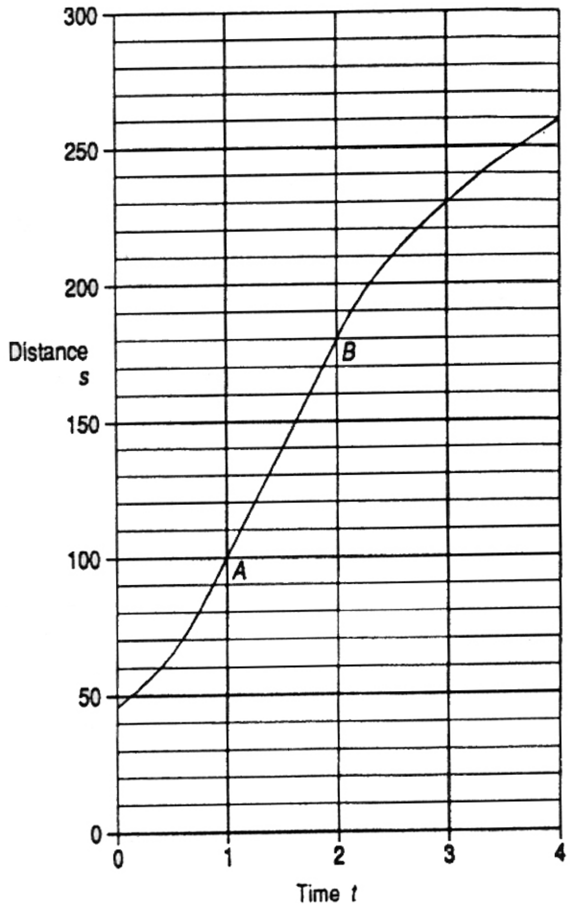
In maths it is common practice to use letters from the first half of the alphabet to represent constants, and letters from the second half to represent variables. The commonest exception to this is the use of n as as constant power (x^n).

The equation of the straight line is written in the form of an equation or a 'formula' for y, so that values of y are worked out from values of x. Because of this, x is called the independent variable and y is called the dependent variable.

The gradient of the line is m.
So if the 'formula' for y is $y = m\,x + c$, the formula for $\frac{dy}{dx}$ is **CURVES AND TANGENTS.**

Well it isn't much of a formula is it? $\frac{dy}{dx} = 3$, or even $\frac{dy}{dx} = m$, these statements don't really give you a sense of exploring the further reaches of higher mathematics. They can hardly be called formulas at all! So let's move on, and try to give $\frac{dy}{dx}$ some dignity.

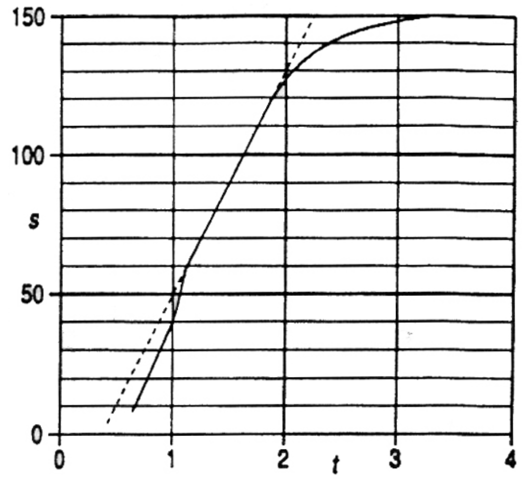
Let us suggest that on a curve, the gradient at any point is the gradient of the tangent at that point.



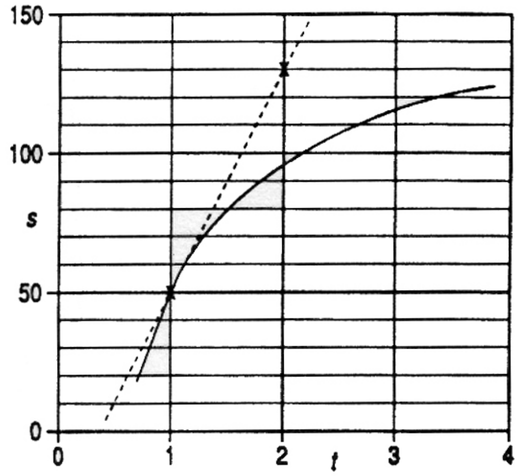
In this graph of distance (miles) against time (hours), the speed is varying all the time except in the region A – B. In that region the graph is a straight line, and the gradient is 80 (the distance goes from 100 - 180 in the second hour), so the speed is 80mph.

But if the graph was describing the motion of a car, you wouldn't say that the car has a speed at any instant could be read from the speedometer. And what does a speed of 80mph mean? It must mean that if the car kept going at the same rate, it would cover 80 miles in 1 hour (a motorist stopped for speeding said 'How could I be doing 80 miles per hour? I've only been driving for 5 minutes!').

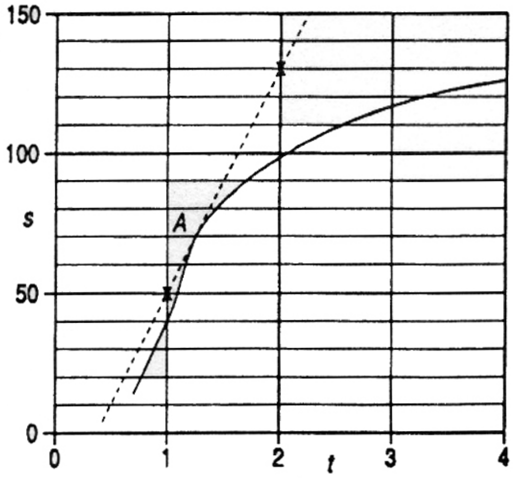
In the graph below, the straight line region is extended as a dashed line to show what would happen if the car kept going at the same speed – 80 miles would be covered in 1 hour. So in the straight line region the car was travelling at 80mph.



We'll now make the straight line region much smaller. The dashed line still represents a change of 80 miles in the second hour.

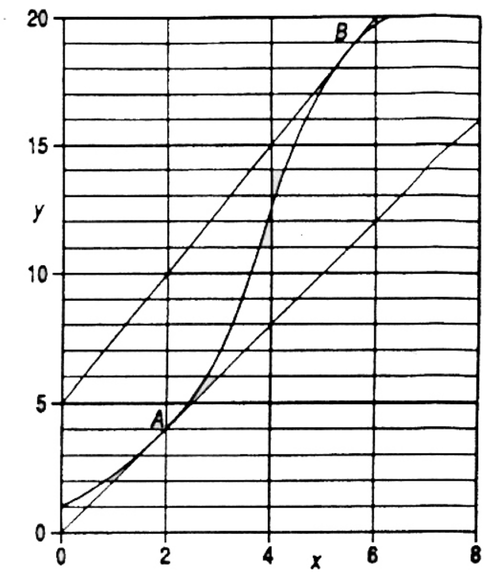


And now it can't be detected as a straight line region at all. But the gradient at A is still the same, and represents a speed of 80mph – that is what the speedometer would say at that moment. The dashed line is a tangent to the curve – that is a line which touches the curve at a point.



So the gradient of a curve at a point is the gradient of the straight line which touches the curve at that point.

Here is a curve with tangents drawn to find the gradients at A and B.



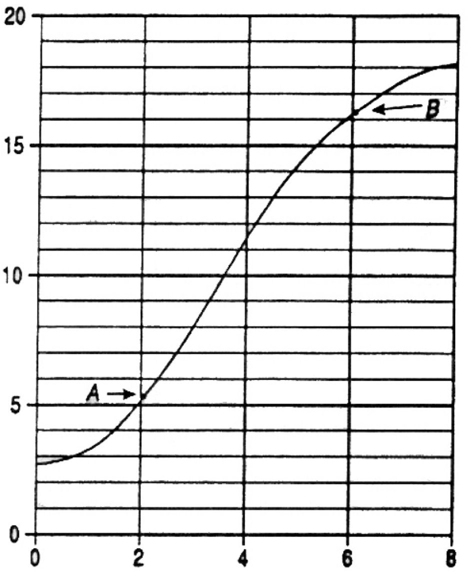
Exercise 9

What are the gradients at A and B?

Now check your answers.

Exercise 10

Here is another curve. Find the gradients at A and B by drawing tangents. (The answers will not be exact, so don't spend too much time on them.



SUMMARY

The gradient of a straight line is $\dfrac{\text{change in vertical variable}}{\text{change in horizontal variable}}$

This is still true when the straight line is just part of a more complicated graph - even a very small part.

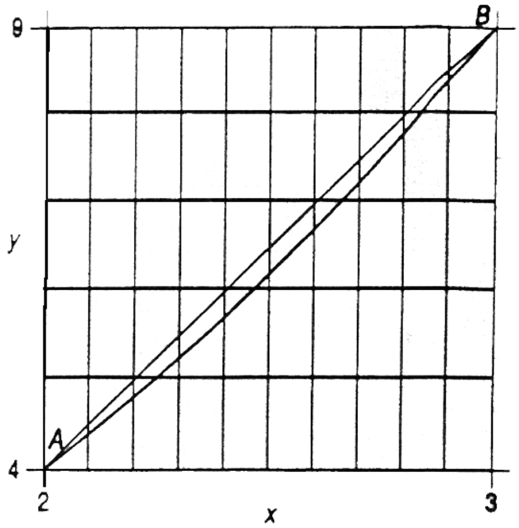
A graph still has gradients even in regions where there are no straight lines.

The gradient of a graph at a point is taken to be the gradient of the tangent at the point. The tangent shows what would happen if the rate of change remained constant. This is like saying that the speed of a car at a certain instant was 80 miles per hour, even though the car travelled only for 5 minutes.

'Normal' graphs (the sort we shall mainly be concerned with) have only one tangent at any point.

CHORDS AND TANGENTS

The diagram shows part of the graph of the curve y = x^2. Let us find the gradient of the curve when x = 2.

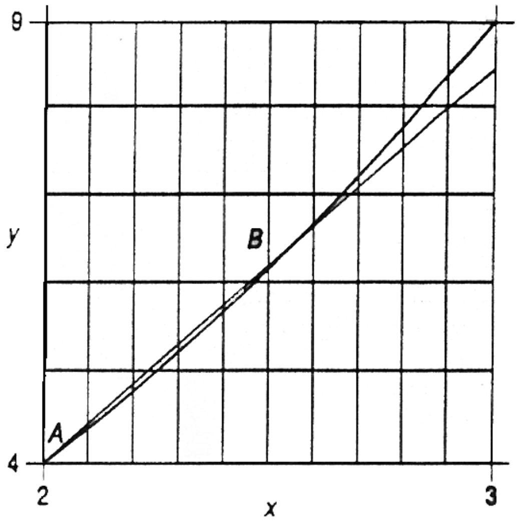


The straight line is a chord which cuts the curve at the end points A (2, 4) and B (3, 9). The gradient of this line is:

$$\frac{(9-4)}{(3-2)} \quad \frac{\text{Change in y}}{\text{Change in } x}$$

In the next diagram the point B has moved to where new x = 2.5, and so new
y = 2.5^2 = 6.25. As point B gets closer to A, the line gets closer to being the tangent at A. The gradient of the line is now

$$\frac{(6.25-4)}{(2.5-2)}=4.5$$



Here is a table of values of new x, new y, and the gradient of the line as B gets closer and closer to A (2, 4).

New x	3	2.5	2.3	2.1	2.05	2.01
New y	9	6.25	5.29	4.41	4.2025	4.0401
Gradient	5	4.5	4.3	4.1	4.05	4.01

Exercise 11

What value do you think the gradient is approaching as B approaches A?

Now check your answer.

Let's see if we can prove the answer.

The point A *has* $x = 2, y = 4.$

Let's suppose that at the point B, x has the value 2 + h.

Then new $y = (2 + h)^2 = 4 + 4h + h^2$

So the increase in y is new y – old y

$= 4 + 4h + h^2 – 4 = 4h + h^2$

Gradient of line = $\frac{increase\ in\ y}{increase\ in\ x} = \frac{4h + h^2}{h} = 4 + h$

This agrees with the table. For example, when x = 2.05 then h must be .05, and the gradient is in fact equal to 4 + .05 = 4.05

The expression for the gradient shows that as h becomes smaller and smaller, the gradient gets closer and closer to 4.

The usual way of stating this mathematically is to say that as h tends to zero, the gradient tends to 4.

SUMMARY

A line which cuts a curve at 2 points called a chord.

As one point approaches the other, the chord becomes more and more like a tangent.

One way of finding the gradient of a curve at a point is first to find the gradient of a chord at that point. As the second point of the chord approaches the first, the value of the gradient of the chord changes, and it may be easy to guess what the final value will be.

ANSWERS

Exercise 1

Pounds per square inch per thousand feet?
Millimetres of mercury per kilometre?
Or, if you're familiar with SI units, Newtons per square metre per metre, which is also Pascals per meter.
Any unit of pressure per any unit of distance will do.

Now return to the text.

Exercise 2

The answer is $s = 30t$.

If you wrote $y = 30t$ then you're on the right lines, but don't forget that variable names other than x and y are common.

If you wrote $t = 30s$ then compare the s and t axes with the more usual y and x axes, to see which corresponds to which. Or just check the values of s when t is 2, and ask yourself what is 30 times what?

Remember $y = mx + c$. This is the equation of a straight line. m is the gradient of the straight line, and c is the intercept on the *y axis*.

In this case the line goes through the origin, so the intercept c is zero.

The gradient is the rate of change of y compared with x (or in this case of s compared with t). If you're not sure about this then you will need to revise it before continuing with this pack.

Now return to the text.

Exercise 3

The gradient of the straight line is 30, corresponding to a speed of 30 miles per hour. s changes 30 times as fast as t, so that when t changes by one hour, s changes by 30 miles.

Now return to the text.

Exercise 4

The equation is $s = 30t + 20$.

If you're not happy about that, compare it with $y = mx +c$, m is the gradient and c is the intercept on the y axis. In this case we have the s axis instead of the y axis, so the constant 'c' is read off from where the line cuts the s axis.

Now return to the text.

Exercise 5

The gradient is 30 (speed 30 miles per hour). It's the same car on the same journey, but this time the graph shows the distance from Coventry. The graph starts when the car is a Watford Gap, so the distance is already 20 miles. But the gradient, which is the speed, is unchanged at 30 miles per hour.

Now return to the text.

Exercise 6

a) 7 b) 7 c) -2 d) 0.3 e) $\frac{2}{3}$ f) -0.24

Now return to the text.

Exercise 7

Yes, this is like the last task, but now we're writing down $\frac{dy}{dx}=\ldots$

a) $\frac{dy}{dx}=2$ b) $\frac{dy}{dx}=5.3$

c) $\frac{dy}{dx}=-5$ d) $\frac{dy}{dx}=6$

e) $\frac{dy}{dx}=\frac{3}{8}$ f) $\frac{dy}{dx}=a$

Now return to the text.

Exercise 8

We can't be certain because it is not stated. But it would be reasonable to guess as follows:

a) *a, b*, and *c* are constants, x and *y* are variables.

b) *g* and *a* are constants, *t* and *v* are variables.

c) *a* is a constant, x and *y* are variables

d) *m* and *n* are constants, *t* and *r* are variables.

e) *k* and *n* are constants, *t* and *r* are variables

f) *a* is a constant, x, *y* and *z* are variables.

Now return to the text.

Exercise 9

At *A* the gradient is $\frac{16}{8} = 2$

At *B* the gradient is $\frac{15}{6} = 2.5$

For A, did you use the intervals 0 to 8 on the x axis, and 0 to 16 on the y axis? And for B, 0 to 6 on the x axis, and 5 to 20 on the y axis? Good! Make the intervals as large as possible to reduce the errors.

If you got 4 and 5, that is probably because you just counted the graduations lines, and forgot that on the x axis each graduation counts as 2.

Now return to the text.

Exercise 10

The gradient at A is the range 2.3 to 2.7.

The gradient a B is in the range 1.3 to 1.6.

Now return to the text.

Exercise 11

Yes! If you didn't get 4, you must be looking for complications that aren't there.

Now return to the text.