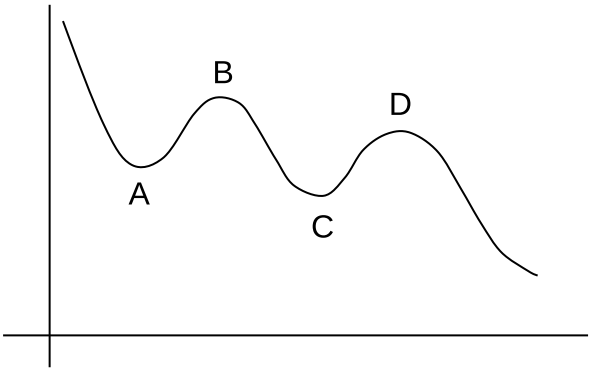
Learning Development

University of Northampton

MAXIMUM AND MINIMUM 1

UPS AND DOWNS



This could represent part of a roller-coaster (straightened out). It would represent the ups and downs of a learner pilot.

It could be the state of morale of somebody on a maths course.

It could represent somebody's sexual activity (no names please!)

Exercise 1

Suggest something else the graph could represent.

Now check your answer.

Whatever the graph represents, the highest point, the maximum value, is at the beginning, and the lowest point, the minimum value, is at the end (notice we have assumed that the beginning is to the left, the end if to the right, and you are not reading this upside down. These are things that most of us would assume, but we have to remain aware that they are assumptions).

Between the 'absolute' maximum and minimum there are two local' maximums, at *B* and *D*, and two 'local' minimums at *A* and *C* (We use words like 'maximum', 'minimum', 'formula' as English words, and therefore put 's' on the end to make the plural. If you prefer 'maxima' etc. then you can alter the text if you like).

As well as the absolute minimum and maximum, the local minimums and maximums are often of interest. Check the list of possible graph topics, and I hope you will agree!

IN BETWEENS

At points like *A, B, C* and *D* the graph is described as 'stationary' because it isn't going up or down. (The way to remember to spell it 'ary' rather than 'ery' is to say 'You have to be wary if it's stationary!')

The question is, 'What's special about these stationary points that allows them to be found?' And since you have recently done pack MC1 (Calculus – Rates of Change), you are probably thinking that the answer has something to do with gradient.

Exercise 2

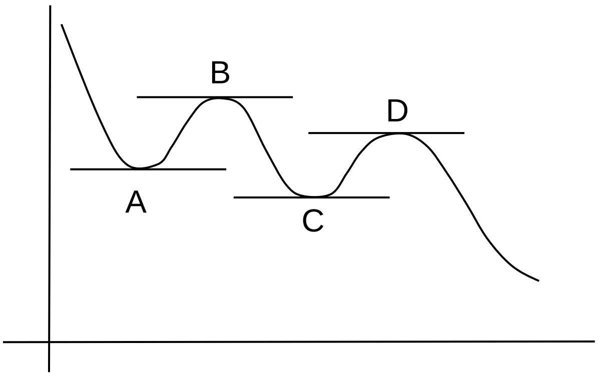
What is the gradient of the graph at points A, B, C and D?

Now check your answer.

The gradient of the graph is the gradient of the tangent. The tangents at these points are all 'horizontal' (parallel to the x axis); and if a line is parallel to the x axis, every point on it has the same y co-ordinate; so

$$\text{Gradient} = \frac{\text{change in y}}{\text{change in } x} = 0$$

In the case of a distance-time graph, for example, the distance is neither decreasing nor increasing at one of these points, so the speed must be zero. Here is the graph with the tangents drawn in.



To find the stationary points by calculus the first need is to have a formula for y. That (usually) makes it possible to find the formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$. Then it should be possible to find the values of x which make $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to zero.

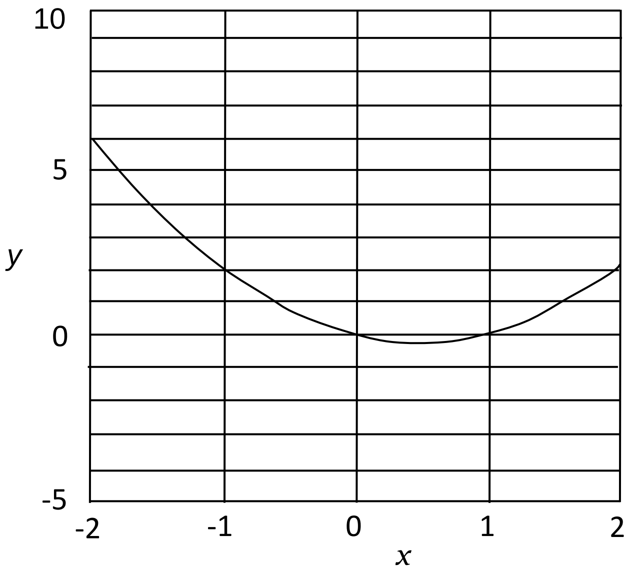
Example

Try $y = x^2 - x$

Here are some values of x and y:

x	-2	-1	0	1	2
y	6	2	0	0	2

It is obvious that the graph has a minimum when x is between 0 and 1; and it looks as though a good guess would be $x = 0.5$.



Try these values of x:

x	0.4	0.5	0.6
y	-0.24	-0.25	-0.24

This is fairly convincing that the minimum point on the graph is the point (0.5, - 0.25) – and done without calculus, as is always possible. But using calculus usually makes it easier!

Exercise 3

$y = x^2 - x$

Write down the formula for $\frac{dy}{dx}$.

Now check your answer.

So the value of x which makes $\frac{dy}{dx}$ equal to zero is $x = 0.5$.

And the corresponding value of y is $(0.5)^2 - 0.5 = -0.25$.

That little exercise contains quite a bit of what calculus is about. If you feel happy with that, then you are now a calculus expert. If you don't feel happy with it, go over it again, and if necessary write down some questions to ask your tutor.

SUMMARY

Between ups and downs, graphs have peaks and troughs which are called 'stationary points' (because the graph isn't going up or down).

The peaks and troughs are 'local' maximums and minimums.

A particular maximum or minimum may not be an absolute maximum or minimum of the graph, but it may still be a point of interest or even mystery ('You have to be wary – if it's stationary').

The maximum and minimum points can always be found by trial and error. But if there is a formula for y, it is usually easier to find them by calculus.

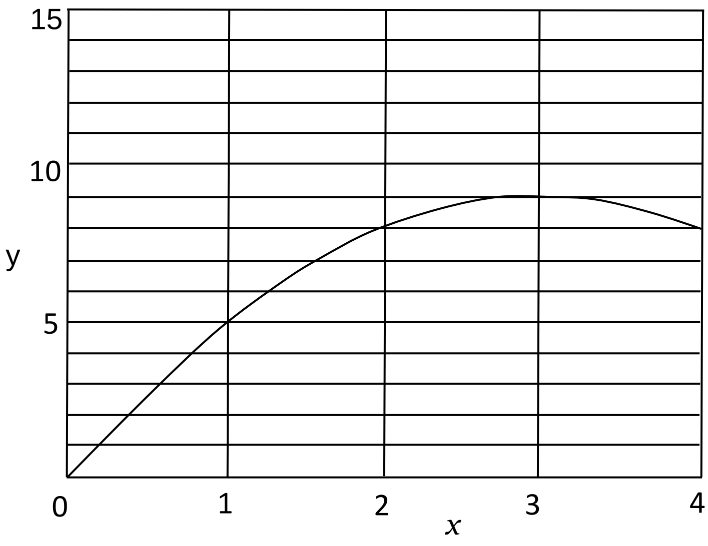
At the maximum and minimum points the gradient of the graph is zero.

The maximum and minimum points can be found by finding the formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and then finding the value of x which makes $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to zero.

A FEW QUADRATICS

Take the equation: $y = 6x - x^2$

The graph is shown below.



To find the stationary point:

$$\frac{dy}{dx} = 6 - 2x$$

So $\frac{dy}{dx} = 0$ when $6 - 2x = 0$, that is when $x = 3$

when $x = 3$ $y = (6 \times 3) - 3^2 = 9$

So the stationary point is the point (3,9).

A number is split into two parts. The two parts are multiplied together. How should the split be made to make the product as large as possible?

For example, take the number 10. Split it into 8 and 2. The product is 16,.7 and 3? The product is 21. 6 and 4? The product is 24. How was this calculated?

Call the number a (this is now a **constant**). If one split is x, the other must be $(a-x)$. The product $y = x(a-x)$.

So $\quad y = ax - x^2$

$$\frac{dy}{dx} = a - 2x$$

So $\frac{dy}{dx} = 0$ when $a - 2x = 0$ giving $x = \frac{a}{2}$ and may y is $\frac{a^2}{4}$.

This shows that the product of the two parts of a given number has a stationary value when the number is split exactly in half. It has not shown that the product is a **maximum**, but a little bit of trial and error will establish that.

A similar calculation will show that the maximum rectangular area that can be enclosed by a given loop of string is a square (proving that the absolute shape is a circle is much harder).

Exercise 4

Find the stationary point in each of the following cases:

a) $\quad y = x^2 - 4x$

b) $\quad y = x^2 - 4x + 7$

c) $\quad y = 3x^2 - 5x - 9$

d) $\quad y = 5 - 12x - 2x^2$

e) The height h of an object thrown vertically upwards is given by $h = ut - \frac{1}{2}gt^2$ (if air resistance is ignored), where u is the throwing speed in m/s, g is the acceleration due to gravity in m/s^2, and t is the time in seconds. If the throwing speed is 30 m/s (a very strong throw), and g is 9.8 m/s^2, find the maximum height.

(Note - this is a case where, by convention, one of the constants, u, comes from the second half of the alphabet).

(Do not forget, in this case, it is $\frac{dh}{dt}$ not $\frac{dy}{dx}$)

CUBICS

Example

$y = 2x^3 - 9x^2 + 12x + 1$

First find $\frac{dy}{dx}$.

$\frac{dy}{dx} = 6x^2 - 18x + 12$

so $\frac{dy}{dx} = 0$ when $6x^2 - 18x + 12 = 0$

The calculus part is finished. What follows is algebra - solving quadratic equations. If you cannot do that, then you need to revise the subject.

$6x^2 - 18x + 12 = 0$

Divide by 6 $\quad x^2 - 3x + 2$

Factorise $\quad (x-1)(x-2) = 0$ (or do it by formula).

$x = 1$ or $x = 2$

When $x = 1$, $y = (2 \times 1^3) - (9 \times 1^2) + (12 \times 1) + 1 = 6$

When $x = 2$, $y = (2 \times 2^3) - (9 \times 2^2) + (12 \times 2) + 1 = 5$

So, there are now **two** stationary points: the points (1, 6) and (2, 5).

Exercise 5

Find the stationary point on the following curves:

a) $y = 2x^3 - 3x^2 - 12x + 8$

b) $y = x^3 - 4x^2 + 5x - 3$

OTHER POWERS

What about this:

$y = x^4 - 4x^3 - 2x^2 + 12 + 5$

$\frac{dy}{dx} = 4x^3 - 12x^2 - 4x + 12 = 0$

So $\frac{dy}{dx}$ = 0 when $4x^3 - 12x^2 - 4x + 12$ = 0

That is the calculus done! All that is left is the algebra. Solving cubic equations is not in the syllabus. You probably will never have to learn to solve cubic equations, because it is easier to use a maths package on the computer. So that means you will not be asked questions like that during this course unless you are given one of the roots. Remember, the calculus part was easy – it's the algebra that takes the time.

The following are examples of when you are given one of the roots.

Example 1

The curve $y = x^4 - 4x^3 - 2x^2 + 12 + 5$ has a stationary point value at the point (1, 12). Find the other two stationary points.

The calculus part of the question is already done. The curve has stationary points at those values x of such that

$$\frac{dy}{dx} = 4x^3 - 12x^2 - 4x + 12 = 0$$

The rest of the answer draws upon the skills you have learnt in others parts of the syllabus. If you do not understand any of the following steps, please revise the relevant pack, as it is not calculus. Since one of the stationary points is (1, 12), the expression for $\frac{dy}{dx}$ must be zero when x = 1. Therefore $(x-1)$ is a factor.

First divide by 4 (to make things easier), then extract the factor $(x-1)$:

$$x^3 - 3x^2 - x + 3 = (x-1)(x^2 - 2x - 3) = 0$$

or divide $x^3 - 3x^2 - x + 3$ by $(x-1)$ by long division, to give $x^2 - 2x - 3$.

The remaining values of x are therefore given by

$$x^2 - 2x - 3 = 0 \qquad (x-3)(x-1) = 0 \quad \text{(or do it by formula)}$$

So x = 3 or x = -1 and the stationary points are (3, -4) and (-1, -4) (y values obtained by putting the x values in the original equations).

Let's review the results.

The problem is to find the stationary points of the curve

$$y = x^4 - 4x^3 - 2x^2 - 12x + 5$$

First find the formula for $\frac{dy}{dx}$

$$\frac{dy}{dx} = 4x^3 - 12x^2 - 4x + 12$$

What values of x make $\frac{dy}{dx}$ equal to zero?

One value is given. x = 1 makes $\frac{dy}{dx}$ equal to zero (check it!)

The other values of x are -1 and 3 (check them!) (finding those values was the hardest part).

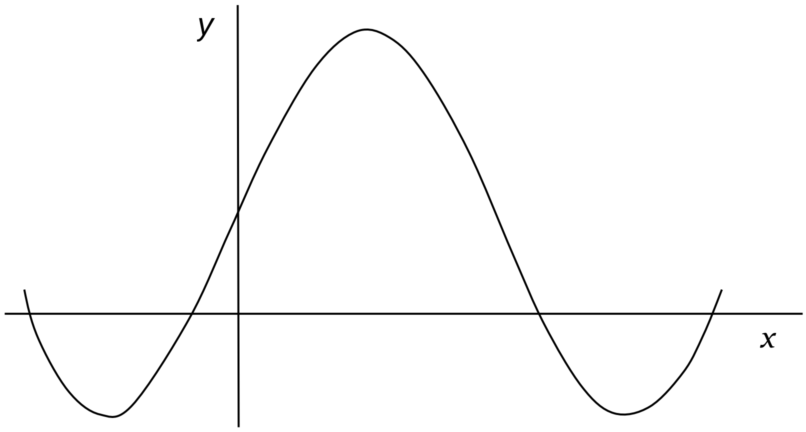
Using the formula for y, x = 1 gives y = 12 (that was given)

x = -1 gives y = -4

x = 3 gives y = -4

So, the points are (1, -12), (-1, -4), (3, -4).

And here is a sketch of the graph of $y = x^4 - 4x^3 - 2x^2 - 12x + 5$.



Example 2

Find the stationary points of $y = x^6 - 3x^2$

First find the formula for $\frac{dy}{dx}$

$$\frac{dy}{dx} = 6x^5 - 6x = 6x\left(x^4 - 1\right)$$

So $\frac{dy}{dx}$ = 0 when $6x\left(x^4 - 1\right)$ = 0

That is the calculus done. Could you write down the values of x?

From the last equation, either x = 0 or $x^4 - 1$ = 0.

If $x^4 - 1$ = 0 then x^4 = 1 so x = 1 or -1 (do not forget the -1).

Using the equation for y, the points work out to be (-1, -2) (0, 0) (1, -2)

Here are a few hints/ reminders about solving equations with powers of x higher than x^2:

a) you have got the expression for $\frac{dy}{dx}$ and put it equal to zero.

Now divide through by any numerical factor, as shown in previous examples.
For example:

$\frac{dy}{dx}$ = $4x^2 - 12x + 8$

Put $4x^2 - 12x + 8$ = 0; divide by 4, $x^2 - 3x + 2$ = 0

This can be done only after equating it to zero. It works because zero divided by any number is still zero.

$\frac{dy}{dx}$ still contains the factor 4. $\frac{dy}{dx}$ = 4 ($x^2 - 3x + 2$)

b) use any solutions you are given. If you know that x = 2 is a solution, the expression must have $(x-2)$ as a factor; so factorise it, or divide by $(x-2)$

c) look out for x or powers of x as factors. For example:

$x^4 + x^3 - 2x^2$ = 0

x^2 is a factor: $x^2\left(x^2 + x - 2\right)$ = 0

so either x^2 = 0 giving x = 0

or $\left(x^2 + x - 2\right)$ = 0 giving x = -2 or 1

Another example: $x^2 - 3x$ = 0

x is a factor: $x\left(x - 3\right)$ = 0

so either x= 0 or x = 3

d) Look out for the cases like $x^3 - 1$ = 0 (x = ±1)

or $x^5 + 32$ = 0 (x = -2)

or $x^2 - 9x$ = 0 (x = ±3)

Exercise 6

a) The curve $y = 3x^4 - 8x^3 - 6x^2 + 24x - 9$ has a stationary value at the point (2, -1).

Find the other two stationary points.

b) Find the stationary points of the curve $y = x^4 - 4x^3 - 8x^2 + 1$

Now check your answers.

SUMMARY

When finding the stationary points of a curve the calculus part consists of finding the formula for $\frac{dy}{dx}$ and asking what values of x make the formula equal to zero.

The resulting equation may be a straightforward linear or quadratic equation, or it may be more complex.

During this course, it will always be possible (by taking out a known factor or other means) to reduce the problem to a linear or quadratic equation or to a simple root (for example if $x^4 - 16 = 0$ then $x^4 = 16$ and $x = \sqrt[4]{16} = \pm 2$).

MAXIMUM OR MINIMUM?

As you see, it is possible to find the stationary point of a curve without drawing a graph. With calculus you can do it without knowing the shape of the graph, or whether the point is a maximum or a minimum. But, in fact, you will often need to know.

There are a number of ways of finding out whether a stationary point is a maximum or a minimum. For example, if y is given by a quadratic with a positive x^2 term, you probably already know the shape of the graph and it has a single stationary point, which is a minimum. If the expression is more complicated, you might find out by plotting a few points. This pack will cover just two ways of finding out.

A LITTLE LESS, A LITTLE MORE.

The first way, try values of x a little bit different from the stationary value.

The equation is $y = x^2 - x$.

Then $\frac{dy}{dx} = 2x - 1$, and so $\frac{dy}{dx}$ is zero when $x = 0.5$

When $x = 0.5$, $y = -0.25$

When $x = 0.4$, $y = -0.24$

When $x = 0.6$, $y = -0.24$

That shows that the stationary point (0.5, -0.25) is a minimum.

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Let's take the cubic, $y = 2x^3 - 9x^2 + 12x + 1$

One of the stationary points was the point (1,6).

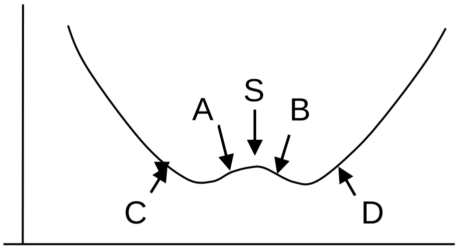
Exercise 7

Find out whether there is a minimum or a maximum by finding y when x is 0.5 and 1.5 (use your calculator if necessary).

Now check your answer.

Just how close to the stationary point should the values of x be?

The first need not go beyond any other nearby maximums and minimums. The points A and B would be okay for testing the stationary point S, but points C and D would give the wrong answer.



With polynomials (collections of positive powers of x) that requirement is sufficient but with some functions, it might be necessary to check more carefully.

Exercise 8

Show that $y = x^4 - 5x^2 + 2$ has a stationary value of $x = 0$.

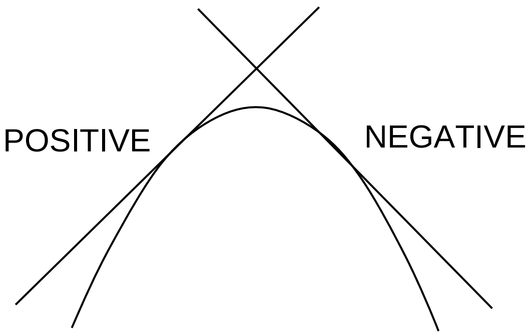
Find whether it is a maximum or a minimum.

Now check your answer.

SLOPING ON AND SLOPING OFF

The second method is to check the gradient $\frac{dy}{dx}$ on each side of the stationary point.

If the gradient is positive just before the point and negative just after, that indicates a **maximum**, as the diagram shows. So, the curve **climbs up** (positive gradient) to a maximum and **falls away** (negative gradient) front it.



And the curve falls into (negative gradient) a minimum and then climbs away (positive gradient) from it.

Look at $y = x^2 - x$ again.

$$\frac{dy}{dx} = 2x - 1$$

So $\frac{dy}{dx}$ = 0 when x = 0.5

To indicate that x is just less or just greater than 0.5, you can write x = 0.5- or x = 0.5+

When x = 0.5- (a little less than 0.5), $2x$ is a little less than 1, and $\frac{dy}{dx}$ is negative.

When x = 0.5+, $\frac{dy}{dx}$ is positive.

So the curve is falling into the stationary point, and climbing away from it. The stationary point is a minimum.

Here is the cubic of Part B again: $y = 2x^3 - 9x^2 + 12x + 1$

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x-1)(x-2)$$

Take the stationary point (1,6) again. When x = 1-, (x-1) is negative, and $\frac{dy}{dx}$ is (negative times negative) which is positive.

When x= 1+, (x-1) is positive, and $\frac{dy}{dx}$ is (positive times negative) which is negative.

So if the curve climbs up to the stationary point and falls away from it - the point is a **maximum.**

When $\frac{dy}{dx}$ can be expressed as a set of factors like that, it makes this method very easy.

Exercise 9

a) The curve $y = x^4 - 4x^3 - 2x^2 + 12x + 5$ has stationary values when x = -1, x = 1, x = 3

Use the 'gradient' method to find whether each point is a maximum or a minimum. (The value of x given here tells you what the factors are)

b) Find the two stationary points of the curve

$$y = 2x^3 - 24x + 20$$

Determine whether each point is a maximum or a minimum.

c) Find the stationary point of the curve $y = \dfrac{3}{x^2} + \dfrac{x}{7}$

Determine whether it is a maximum or a minimum.
(Hint: it can be written as $y = 3x^{-2} + \dfrac{1}{7}x$)

Now check your answers.

SUMMARY

You often need to know whether a stationary point is a maximum or a minimum.

One way of finding out is to find what happens to the graph just before and after the stationary point. Another way is to find what happens to the gradient just before and just after the stationary point.

If the graph climbs up to the point (gradient positive) and then falls away from it (gradient negative), the point is a maximum.

If the graph falls down to the point (gradient negative) and climbs away (gradient positive), the point is a minimum.

ANSWERS

Exercise 1

The Financial Times 100 Share Index. Mother Demdike's haemogoblin count. The Prime Minister's popularity and so on and so on.

Now return to the text.

Exercise 2

The gradient of the graph at these points is zero.

Now return to the text

Exercise 3

$$\frac{dy}{dx} = 2x - 1$$

If you wrote down $\frac{dy}{dx} = 2x$, then you are probably a victim of mathematical notation. Since there is nothing in front of the x term, and no power, then differentiating must give nothing! But $x^2 - x$ is really $1x^2 - 1x^1$. It is the mathematical convention that all the ones are left out: gone but not forgotten! (not in future!).

Now return to the text.

Exercise 4

a) $y = x^2 - 4x \quad \frac{dy}{dx} = 2x - 4 \quad \frac{dy}{dx} = 0$ when $x = 2$

When $x = 2 \quad y = 2^2 - 4 \times 2 = -4$

So the stationary point is (2, -4)

If you did not get the answer, follow the working through carefully. Find out exactly where your difficulty is. You will probably find that you can then overcome it. If you cannot, write down a question to ask the maths tutor. If you can, try the other problems.

b) $\frac{dy}{dx} = 2x - 4$, just as in a). $\frac{dy}{dx} = 0$ when $x = 2$

When $x = 2 \quad y = 2^2 - 4 \times 2 + 7 = 3$

The point is (2,3)

c) $\frac{dy}{dx} = 6x - 5$ $\frac{dy}{dx}$ = 0 when $6x - 5$ = 0, $x = \frac{5}{6}$

When $x = \frac{5}{6}$ $y = 3\left(\frac{5}{6}\right)^2 - 5\left(\frac{5}{6}\right) - 9 = -11\frac{1}{2}$

The point is $\left(\frac{5}{6}, -11\frac{1}{2}\right)$

Or you might have (0.833, -11.083).

d) $\frac{dy}{dx} = -12 - 4x$ The point is (-3, 23)

e) The answer is 45.92m. This problem needs several steps.

It is probably easier to put the numbers in immediately.

$h = 30t - \frac{1}{2} \times 9.8 \times t^2 = 30t - 4.9t^2$

$\frac{dh}{dt} = 30 - 9.8t$

$\frac{dh}{dt}$ = 0 when $30 - 9.8t$ = 0, $t = \frac{30}{9.8}$ = 3.061

So max $h = 30 \times 3.061 - 4.9 \times (3.061)^2$ = 45.92

The maximum height reached is 45.92m.

Now return to the text.

Exercise 5

a) The stationary points are (-1, 15) and (2, -12). If you got those first time, then you are forging ahead! There are quite a few traps for the unwary.

$y = 2x^3 - 3x^2 - 12x + 8$

$\frac{dy}{dx} = 6x^2 - 6x - 12$ $\frac{dy}{dx}$ = 0 when $6x^2 - 6x - 12$ = 0

Look out for numeric factors! Divide through by 6.

$x^2 - x - 2$ = 0 (x + 1) (x -2) = 0

x = -1 or 2 (or do it by formula)

It is quite easy to make mistakes when finding y.

x = -1, y = $2(-1)^3 - 3(-1)^2 - 12(-1) + 8$

= $2 \times (-1) - 3 \times (1) - 12 \times (-1) + 8$

= -2 -3 + 12 + 8 = 15

x = 2, y = $2(2)^3 - 3(2)^2 - 12(2) + 8$

= 16 - 12 - 24 + 8 = -12

b) The points are (1, -1) and $\left(\frac{5}{3}, -1\frac{4}{27}\right)$ Or (1.67, -1.15)

This is another case where finding the values of x is easier than finding the values of y.

$y = x^3 - 4x^2 + 5x - 3 \quad \frac{dy}{dx} = 3x^2 - 8x + 5$

$\frac{dy}{dx}$ = 0 when $3x^2 - 8x + 5$ = 0

$(x-1)(3x-5)$ = 0

x = 1 or $\frac{5}{3}$ (or do it by formula)

When x = 1, y = 1 – 4 + 5 – 3 = -1

When $x = \frac{5}{3}$, $y = \left(\frac{5}{3}\right)^3 - 4\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - 3$

$= \frac{125}{27} - 4\left(\frac{25}{9}\right) + \frac{25}{3} - 3$

$= \frac{-31}{27}$

A rather tedious calculation, you might prefer to use your calculator. To avoid loss of accuracy, store the result of $\frac{5}{3}$ in the memory for use in the equation. If you reduce the decimal places too early, quite a large error can accumulate.

Exercise 6

a) The stationary points are (-1, -28) and (1,4).

If you have only the x values correct (x = ±1) then you have done the calculus correctly, and solved the equation for x. Try the calculations for y again.

$y = 3x^4 - 8x^3 - 6x^2 + 24x - 9$

$\frac{dy}{dx} = 12x^3 - 24x^2 - 12x + 24$

$\frac{dy}{dx}$ = 0 when $12x^3 - 24x^2 - 12x + 24$ = 0

Divide by 12: $x^3 - 2x^2 - x + 2$ = 0

One stationary point is (2, -1), so x = 2 must be one solution. That means (x - 2) is a factor.

Factorise (or divide by (x - 2)):

$x^3 - 2x^2 - x + 2 = (x-2)(x^2-1)$ = 0

So either $x - 2$ = 0 (which we know) or $x^2 - 1$ = 0

What are the other values of x?

Don't forget the -1, x = ±1

When x = 1, y = 4. When x = -1, y = -28

The points are (1, 4) and (-1, -28).

b) The stationary points are (-1, -2), (0, 1) and (4, -127).

$y = x^4 - 4x^3 - 8x^2 + 1$

$\frac{dy}{dx} = 4x^3 - 12x^2 - 16x$

$\frac{dy}{dx}$ = 0 when $4x^3 - 12x^2 - 16x$ = 0

Divide by 4 $x^3 - 3x^2 - 4x$ = 0

See factor x? $x(x^2 - 3x - 4)$ = 0

Either x = 0 or $x^2 - 3x - 4 = (x+1)(x-4)$ = 0

So x = 0 or -1 or 4

Using the equation for y, the points are (-1, -2) (0, 1), (4, -127).

Now return to the text.

Exercise 7

It is a maximum.

When x = 0.5 and when x = 1.5, y = 5.5

Now return to the text.

Exercise 8

When x = 0, y is a maximum

If you didn't get that answer, make sure you understand why before you move on. Check through the working to make sure you have not missed anything.

$$y = x^4 - 5x^2 + 2, \ \frac{dy}{dx} = 4x^3 - 10$$

x is a factor (as well as 2)

So $\frac{dy}{dx}$ is zero when x = 0. That proves that y has a stationary value when x = 0.

The other satisfactory values are when $x^2 = \frac{5}{2}$, which gives x somewhere near ±1.5. that means we can safely use x = ±1 to test for maximum or minimum; there is no stationary point as close as or closer than this.

When x = -1, y = -2

When x = 0, y = 2

When x = 1, y = -2

So when x = 0 is a maximum.

Now return to the text.

Exercise 9

a) x = -1 …..minimum

x = 1 ….. maximum

x = 3 ….. minimum

It is important that you get it right using the gradient method.

$$y = x^4 - 4x^3 - 2x^2 + 12x + 5$$

$$\frac{dy}{dx} = 4x^3 - 12x^2 - 4x + 12$$

$$\frac{dy}{dx} = 4(x+1)(x-1)(x-3)$$

When x is near -1, the second and the third brackets are both negative, so are positive when multiplied together. So when x is -1- (a little less than -1, say -1.1) the first bracket is negative, so $\frac{dy}{dx}$ is negative. When x is -1+ (say -0.9), the first bracket is positive, so $\frac{dy}{dx}$ is positive.

This shows that is a minimum when x = -1

When x is near 1, the first bracket is positive and the third bracket is negative, so $\frac{dy}{dx}$ is positive.

When x is 1+, the second bracket is positive, so $\frac{dy}{dx}$ is negative.

This shows that y is a maximum when x = 1.

The process may seem rather long and tedious when written out in full detail; but it can be made quite speedy with practice, and is often the best test.

b) The point (-2, 52) is a maximum.

The point (2, -12) is a minimum.

$$y = 2x^3 - 24x + 20$$

$$\frac{dy}{dx} = 6x^2 - 24$$

$\frac{dy}{dx}$ = 0 when $6x^2 - 24$ = 0

Divide by 6, $x^2 - 4$ = 0, x^2 = 4, x = ±2

$$\frac{dy}{dx} = 6(x^2 - 4) = 6(x+2)(x-2)$$

When x = -2-, both brackets are negative, $\frac{dy}{dx}$ is positive

When x = -2+, the brackets are negative and positive, $\frac{dy}{dx}$ is negative.

The point (-2, 52) is a **maximum**.

When x = 2-, the brackets are positive and negative, $\frac{dy}{dx}$ is negative.

When x = 2+, both brackets are positive, $\frac{dy}{dx}$ is positive.

The point (2, -12) is a **minimum**.

c) The point is (3.476, 0.745) and it is a minimum.

You need to have your wits about you to get that right; so if you have, well done!

To start with, you need to remember that $\frac{3}{x^2}$ is the same as $3x^{-2}$

Then $\frac{dy}{dx} = -6x^{-3} + \frac{1}{7}$

$\frac{dy}{dx} = 0 \quad -6x^{-3} + \frac{1}{7} = 0$

So $-6x^{-3} + \frac{1}{7} = 0 \quad \frac{1}{7} = \frac{6}{x^3}$

Multiply both sides by $7x^3$: $x^3 = 42$ $x = 3.476$ put that value into the equation for y.

$y = \frac{3}{3.476^2} + \frac{3.476}{7} = 0.745$

The point is (3.476, 0.745)

$\frac{dy}{dx} = -\frac{6}{x^3} + \frac{x}{7}$

$\frac{dy}{dx}$ is zero when x is 3.476. When x is 3.476- the first term is slightly larger in size, so $\frac{dy}{dx}$ will be negative.

When x =3.476+, $\frac{dy}{dx}$ will be positive. So the point is a minimum.

Now return to the text.