DERIVATIVES

SINES AND COSINES

The Operator $\frac{\mathrm{d}}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}$ has been much mistreated over the years

In its youth it suffered an immigration ban. Isaac Newton didn't like it because he hadn't invented it; it was invented abroad by a German called Leibniz. Isaac had invented something which did a similar job. English mathematicians followed Isaac; this caused English mathematics to fall behind for a century or so.

Once they started using $\frac{\mathrm{d}y}{\mathrm{d}x}$ mathematicians surrounded it with all sorts of rules to prevent other people (like engineers) using it in a useful manner; "leave it to the professionals" was their advice. "You have to be very careful about continuity", they would say, shaking their heads, or "Don't move an inch without Weirstrass's principle of the point of accumulation", or "$\frac{\mathrm{d}}{\mathrm{d}x}$ is an operator: $\frac{\mathrm{d}y}{\mathrm{d}x}$ cannot be treated as a fraction". In fact $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be treated in almost any way at all, within the normal rules of mathematics, and doesn't seem to mind in the least. The motto is, if it works, do it! Just a word of warning: one thing you can't do is cancel the d. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not the same as $\frac{y}{x}$. The d is not a multiplier; it means 'a very (very) small change in …..' $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be treated in many ways like a fraction; but $\frac{\mathrm{d}}{\mathrm{d}x}$ can also be treated like an 'operator'. That is to say, $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be viewed as the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ operating on y to give the derivative of y: $\frac{\mathrm{d}y}{\mathrm{d}x}$ means $\frac{\mathrm{d}}{\mathrm{d}x}$ (y).

That's rather like saying that 3 x (8) = 24, 3 x (21) = 63 and so on.

This means you can write things like $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$,

or $\frac{\mathrm{d}}{\mathrm{d}x}(2x^5 - 3x^2 + 7x - 8) = 10x^4 - 6x + 7$. So it's another way of saying that you are differentiating, which may be convenient at times.

Try it. Write the answers to the following in the form (for example) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$.

Exercise 1

Differentiate:

1.	$3x^7$	2.	$2x^4 - 3x^3 + 5x^2$
3.	$7x^{-1} - 12x^{-2}$	4.	$3x^{\frac{5}{3}} + 9x^{\frac{2}{3}}$

Now check your answers.

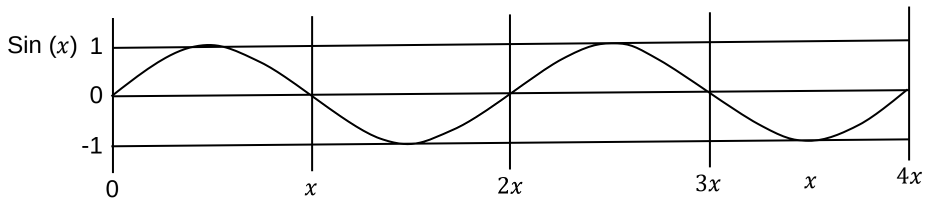
Learning Development

University of Northampton

Sin (x)

So now we're going to differentiate sin (x), or (to put it another way) find $\frac{\mathrm{d}}{\mathrm{d}x}$(sin (x)), or (to put it another way) find f' (x) if f(x) =sin(x), or (to another way) find $\frac{\mathrm{dy}}{\mathrm{d}x}$ if y = sin (x).

Here's a graph of the function sin (x).



First let me remind you that when x is very small, sin (x) is about equal to x (but only if the angle is measured in **radians**. In this Pack the angle will **always** be measured in radians).
Try it on your calculator. First make sure that it's set to 'radians'. Then find $\frac{\sin(x)}{x}$ when x is small, by dividing sin (0.1) by 0.1 (0.1 radians is about 6 degrees). Try smaller values of x. You should find that the ratio gets closer to 1 as x gets smaller.

This suggests that when x is near to zero, differentiating sin (x) gives the same answer as differentiating x.

When $x \rightarrow 0$, $\frac{\mathrm{d}}{\mathrm{d}x}$ (sin(x)) $\rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$.

And when x is $\frac{\pi}{2'}$ sin (x) reaches a maximum of 1 and $\frac{\mathrm{d}}{\mathrm{d}x}$ (sin (x)) is 0.

So as x goes from 0 to $\frac{n}{2}$ the derivative of sin (x) decreases from 1 to 0.

It so happens that cos (x) also decreases from 1 to 0 as x goes from 0 to $\frac{\pi}{2}$. So you might guess that the derivative of sin (x) is cos (x).

You have guessed correctly!

Let's try it 'from first principles'. Put y = sin (x)

Suppose x increases by the small amount h.

Then new y = sin (x + h) = sin (x) cos (h)+ cos (x) sin (h) (this uses the expansion of sin (A + B) given in the trig module.) When h is very small, sin (h) is (near enough) the same as h. What happens to cos (h) when h is very small? Do you remember? When h is very small, cos (h) is (near enough) equal to 1. So when h is very small,
new y = (x) + h cos (x).

So change in y = new y – old y = sin (x) + h cos (x) –sin (x) = h cos (x)

And $\frac{\text{dy}}{\text{d}x}$ = $\frac{\text{change in y}}{\text{change in x}}$ = $\frac{h\cos(x)}{h}$ = cos (x)

Exercise 2

If $f\ (x)$ = sin (x), find

1. $f\ (1)$ 2. $f'(1)$
3. $f\ (0.3)$ 4. $f'(22)$
5. $f'(-1)$

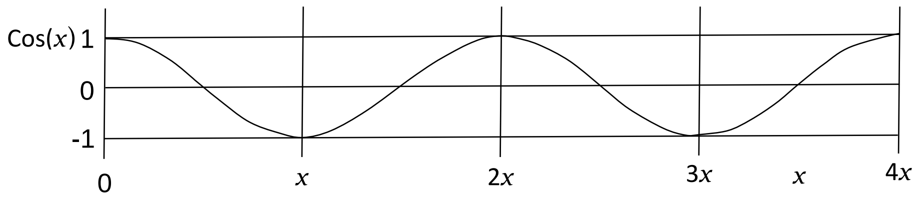
Now check your answers.

Cos (x)

Would you like to guess what you get if you differentiate cos (x)?

If your guess is sin (x) it's a good guess; but its wrong.

Here's a graph of cos (x). As x goes from 0 to $\frac{\pi}{2}$ the gradient is negative, so the derivative can't be sin (x), which is positive in this region.



What about – sin (x)?

$\frac{\text{d}}{\text{d}x}$ (cos(x)) = -sin (x)

Exercise 3

If f (x) = cos (x), find

1. $f\ (0.75)$ 2. $f'(0.75)$
3. $f'(3)$ 4. $f\ (-3.1)$
5. $f'(-0.2)$

Now check your answers.

We are not going to go through it 'from first principles', but you can if you like. It's not very hard as long as you can remember the expansion of cos (A + B).

THE CHAIN RULE

Knowing the derivatives of sin (x) and cos (x) is all very well, but it's not too often you come across these simple functions in real life. You're more likely to see things like 2.7sin (17.3x + 1.89) and other such awkward functions.

Let's first of all consider a straight forward multiplier:

$y = 3 \sin(x)$

You already know how to deal with this. The multiplier stays as a multiplier while the rest is differentiated.

$$\frac{dy}{dx} = 3 \cos(x)$$

And so on: $\frac{d2y}{dx2} = 3 (-\sin(x)) = -3 \sin(x)$

But what about this:

$y = \sin(4x)$

The 4 is not a simple multiplier; it **doesn't** just multiply the function by 4. Something new is needed. And here it is:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is called the 'chain' rule or the 'function' rule. Notice how the derivatives appear to be behaving like fractions: if you cancel the top du with the bottom du on the right-hand-side, you get back to $\frac{dy}{dx}$.

And here's how to use it. I want to find $\frac{dy}{dx}$ if $y = \sin(4x)$

Put $u = 4x$

Now $y = \sin(u)$

(y is now a function of u, which is a function of x. That's where 'function of a function' comes from.)

$$\frac{du}{dx} = 4$$

$$\frac{dy}{du} = \cos(u)$$

So $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ = cos (u) x 4 = 4 cos (u)

But we don't want the answer in terms of u; u was introduced as a temporary measure to help with the working. So remembering that u was defined as 4 x, the final answer is

$$\frac{dy}{dx} = 4\cos(4x)$$

The same method can be used if, for example, y = sin (5 x +7)

Put u = $5x + 7$

Now y = sin (u)

$$\frac{du}{dx} = 5$$

$$\frac{dy}{du} = \cos(u)$$

So $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ = cos (u) x 5 = 5 cos (u)

= 5 cos ($5x$ +7)

Now quite complicated functions can be done with ease!

y = cos ($5x^7 - 4x^3 + 9$)

Put u = $5x^7 - 4x^3 + 9$

y = cos (u)

$$\frac{dy}{du} = -\sin(u)$$

$$\frac{du}{dx} = 35x^6 - 12x^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin(u) \times (35x^6 - 12x^2)$$

$$= -(35x^6 - 12x^2)\sin(5x^7 - 4x^3 + 9)$$

This is not an expression we would want to spend too much time with; but the differentiation is fairly straight forward.

Let's look at the function we started off with:

$$y = 2.7 \sin(17.3x + 1.89)$$

Put $u = 17.3x + 1.89$

$$y = 2.7 \sin(u)$$

$$\frac{dy}{du} = 2.7 \cos(u)$$

$$\frac{du}{dx} = 17.3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2.7 \cos(u) \times 17.3$$

$$= 46.71 \cos(17.3x + 1.89)$$

Finally a slightly different function, $y = \sin^4(x)$ (remember that $\sin^4(x)$ means $(\sin(x))^4$).

Put $u = \sin(x)$

Then $y = u^4$

$$\frac{dy}{du} = 4u^3$$

$$\frac{du}{dx} = \cos(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3\cos(x) = 4\sin^3(x)\cos(x)$$

(The chain rule is normally written without the '$\times$' sign, so that's how it will be written in this pack from now on).

Exercise 4

1. Find $\frac{dy}{dx}$ in the following cases:

a) $y = \sin(6x)$

b) $y = \cos(5x)$

c) $y = 3\sin(5x)$

d) $y = \sin(x^2)$

e) $y = \cos(7x - 3)$

f) $y = 4\cos(7x - 3)$

g) $y = \sin(x^3 - 4x^2 + 7x - 3)$

h) $y = 3\cos(5x^2 - 6x)$

i) $y = \sin(3\cos(x))$

2. $f\,(x)$ = 3sin (2x + 1)

Find $f\,(0)$, $f'(0)$, $f'(-1)$, and $f'(0.5)$

Now check your answers.

AN EXCEPTION

The following should now be familiar to you:

$$\frac{\mathrm{d}\,(x^{n})}{\mathrm{d}x} = \mathrm{n}\,x^{(\mathrm{n}-1)}.$$

$\frac{d\,(x^{0.1}}{dx}$) produces an $x^{-0.9}$, $\frac{d\,(x^{-0.1})}{dx}$ produces an $x^{-1.1}$, but x^{-1} never appears. It would need an x^{0} to produce an x^{-1}; but x^{0} is just a constant 1 (even when x is zero!), which as you know gives 0 when differentiated.

When you differentiate ln(x) (that is, log of x to the base e) you get x^{-1}.

$$\frac{d\,(\ln(x))}{dx} = x^{-1} = \frac{1}{x}$$

To put it another way, if y = ln(x)

Then $\frac{dy}{dx} = \frac{1}{x}$

Example 1

From first principles: y = ln(x)

Suppose x increases by a small amount h then

new y = ln (x + h)

So increase ln y = new y – old y = ln (x=h) – ln(x)

Remember the rules of logs: log (ab) = log (a) + log(b)

log $\left(\frac{a}{b}-\right)$ = log (a) – log (b)

So increase in y = ln (x + h) – ln (x) = ln $\left[\frac{x+h}{x}-\right]$

When z is very small (letter z picked at random), ln (1 + z) is very close to z.
For example:

ln (1+.4) = ln (1.4)
= .336 (19% difference between .336 and .4)

ln (1 + .2) = .18 (11% difference)

1n (1 + .1) = .095 (5% difference)

1n (1 +.05 = .0488 (2.5% difference)

Try some more with your calculator. You will see that as z gets smaller, 1n (1+z) gets closer and closer to z.

It follows that when h gets very small, ln $(1+\frac{h}{x})$ is (near enough) equal to $\frac{h}{x}$.

So with very (very) small h,

$$\frac{\text{change in y}}{\text{change in } x} = h/x = \frac{1}{x}$$

This means that if y = ln (x)

Then $\frac{dy}{dx} = \frac{1}{x}$

Exercise 5

If $f\ (x)$ = ln (x) find:

a) $f\ (5)$ b) $f'(4)$ c) $f\ (1)$

d) $f'(1)$ e) $f'(10)$ f) $f'(0.1)$

(use your calculator as necessary).

Now check your answers.

More

If you can remember and use the basic rules of logs, you can expand the use of this result. The rules are:

ln (ab) = ln (a) + ln (b)

ln $\left(\frac{a}{b}\right)$ = ln (a) – ln (b)

ln (a^n) = n ln (a)

So if y = ln (5 x)
then y = ln (5) + ln (x)

and $\frac{dy}{dx} = \frac{1}{x}$ because ln (5) is a constant, so its derivative is zero.

And if y $= \ln\left(\frac{x}{7}\right)$

then y $= \ln(x) - \ln(7)$

and $\frac{dy}{dx} = \frac{1}{x}$

If y $= \ln(x^4)$

then y $= 4\ln(x)$

and $\frac{dy}{dx} = 4\left(\frac{1}{x}\right) = \frac{4}{x}$

This also applies to negative powers. One which can catch you out if your are not careful is this:

y $= \ln\left(\frac{1}{x}\right)$

This can be written as y = ln (x^{-1}) = (-1) ln (x) = - ln(x)

So $\frac{dy}{dx} = -\frac{1}{x}$

You might find a mixture:

y $= \ln(8x^3)$

y $= \ln(8) + \ln(x^3) = \ln(8) + 3\ln(x)$

and $\frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$

Exercise 6

1. Find $\frac{dy}{dx}$ in the following cases:

a) y = ln $(12x)$ b) y = ln(a x)

c) y = ln (x^3) d) y = ln $(3x^2)$

e) y = ln $\left[\frac{1}{x^3}\right]$ f) y = ln $(x^{\frac{2}{3}})$

2. If y = ln $(3x^2)$ find the value of y and or $\frac{dy}{dx}$ when $x = 5$.

3. Given that $f\ (x)$ = ln (2 x^{-3}) find:

a) $f\ (2)$ b) $f'(2)$ c) $f\ (0.5)$ d) $f'(0.1)$

Now check your answers.

As with sines and cosines, the result can be greatly extended using the chain rule.

Here are some examples:

Example 1

$$y = \ln(x + 3)$$

Put $u = x + 3$

Then $y = \ln(u)$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = 1$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u} = \frac{1}{x+3}$$

Example 2

$$y = \ln(3x + 7)$$

Put $u = 3x + 7$

$$y = \ln(u)$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\,3 = \frac{3}{u} = \frac{3}{(3x+7)}$$

Example 3

$$y = \ln(x^2 - 7x + 5)$$

Put $u = x^2 - 7x + 5$

$$y = \ln(u)$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = 2x - 7$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}(2x - 7)$$

$$= \frac{(2x-7)}{u} = \frac{(2x-7)}{(x^2-7x+5)}$$

Example 4

$$y = \ln(\cos(x))$$

Put $u = \cos(x)$

$$y = \ln(u)$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = -\sin(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}(-\sin(x)) = \frac{-\sin(x)}{u}$$

$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

Example 5

$$y = \ln(x^2)$$

The easiest way is to write this as y = ln(x).

But you could say u = x^2

$$y = \ln(u)$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{u}\,2x = \frac{2x}{u} = \frac{2x}{{}_{x}2} = \frac{2}{x}$$

which is the same result as before.

REMEMBER - logs of negative numbers are outside the scope of this pack. If you find yourself needing the logarithm of a negative number, you have probably made a mistake!

Now try some differentiations.

Exercise 7

1. Find expressions for $\frac{dy}{dx}$ in the following cases:

a) y = ln (x -3) b) y = ln (x +12)

c) y = ln (5 x -9) d) y = ln (x^2+8)

e) y = ln (x^2+7 x) f) y = ln (x^2-3 x +4)

g) y = ln (5 x^3-9) h) y = ln (cos(2 x))

2. a) $f(x)$ = ln(2 x +1) Find $f(1)$ and $f'(2)$

b) $f(x)$ = ln(x^2-5 x +3) Find $f(5)$ and $f'(8)$.

3. Some altimeters depend on the change of atmospheric pressure with height. The height h in metres corresponding to an atmospheric pressure p is given approximately by:-

$$h = H \ln\left(\frac{G}{p}\right)$$

where H = 7000 and G is the atmospheric pressure at ground level.

a) Write down the formula for $\frac{dh}{dp}$
(the height change per unit change of pressure)

If the ground level pressure is 760 mm of mercury, find

b) the change of height for 1 mm change of pressure at ground level (p = G)

c) the height at which the pressure is 76 mm of mercury

d) the change of height for 1 mm change of pressure at this height.

Now check your answers.

SUMMARY

The derivative of ln (x) is x^{-1} or $\frac{1}{x}$.

The rules of logarithms can be used to differentiate such functions as

ln (3 x) (answer: $\frac{1}{x}$) and ln (x^5) (answer: $\frac{5}{x}$)

The chain rule can be used to differentiate such functions as

ln(3 x +7) (answer: $\frac{3}{3x+7)}$) and in (cos (x)) (answer:-tan(x)).

DERIVATIVE OF e^x

The number e, which is approximately 2.71828, has many properties. One of them is that powers of e can be found by using a series, rather than going through a complicated process of powers and roots (for example, to find $5^{3.25}$, you will have to cube the number 5 and find the fourth root). This is why e was used as the base of early logarithms (Napierian logarithms, which is now often called the natural logarithms).

The series is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^r}{r!} + \cdots$$

and so on to infinity.

You may not have seen 2!, 3!, and so on before. They are called 'factorials', e.g.
3 x 2 x 1 (= 6), 4! means 4 x 3 x 2 x 1 (= 24), and so on.

What we're concerned with at the moment is derivatives; so I'm going to differentiate ^{x}, term by term,

$$\frac{d}{dx}(e^x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \frac{rx^{r-1}}{r!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^{r-1}}{(r-1)} + \cdots$$

The fact that the first term has disappeared doesn't matter, because this is the sum to infinity (in practice, of course, the sum until the terms become so small that they make no difference).
This means that $\frac{d(e^x)}{dx} = e^x$

The function is equal to its own derivative!

The result:

$$y \ = e^x$$

$$\frac{dy}{dx} \ = \ e^x$$

Exercise 8

If $f\,(x) = e^x$, calculate

a) $f(0)$ b) $f'(0)$ c) $f\,(1.5)$ d) $f'(1.5)$

Now check your answers.

Here is another trick. If it helps, good! If it does not, just carry on.

If $y = \ e^x$

Then, $x =$ ln (y) (this comes from the definitions of powers and logs)

Now differentiate with respect to y :

$$\frac{dy}{dx} = \frac{1}{y} = \frac{1}{e^x}$$

Just as with ordinary fractions, both sides can be turned upside down:

$$\frac{dy}{dx} = \frac{e^x}{1} = e^x$$

OTHER EXPONENTIALS

The rules of logarithms were quite extending the derivatives of ln (x) but the rules of powers are not very helpful in extending derivatives of e^x. For example,

$$y = e^{x+5}$$

This can be written as $y = e^x e^5$ or $y = e^5 e^x$, so e^5 is just a multiplying constant, and

$\frac{dy}{dx} = e^5 e^x$ (the constant times the derivative of e^x) = e^{x+5}

This is just the same as $y = Ae^x$, where *A* is a constant, so

$\frac{dy}{dx} = Ae^x$ (the constant times the derivative of e^x)

So, there is a whole family of functions (namely, Ae^x) whose derivatives are the same as the original function.

The chain rule is much more helpful. Here are some examples:

a) $y = e^{3x}$

Put $u = 3x$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = 3$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u 3 = 3e^u = 3e^{3x}$

b) $y = e^{x^2}$ (this means e to the power (x^2), not e^x squared, which would have to be written as $(e^x)^2$).

Put $y = x^2$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\, 2x = 2x\, e^u = 2xe^{x^2}$

c) $y = e^{\sin(x)}$

Put $u = \sin(x)$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

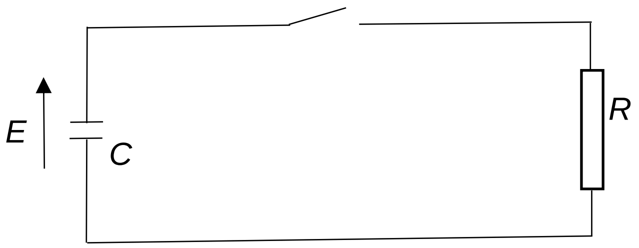
$$\frac{du}{dx} = \cos(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = e^u\cos(x) = \cos(x)e^{\sin(x)}$$

d) When a capacity C is discharging through a resistor R, the voltage v on the capacitor is given by

$$v = Ee^{\left(-t/_{CR}\right)}$$

Where E is the initial voltage on the capacitor (when t = 0), and t is the time in seconds. (see diagram)



Switch closes at time t = 0

The rate of charge in volts per second is given by

$$\frac{dv}{dt} = Ee^{\left(-t/_{CR}\right)}\frac{-1}{CR} = -\frac{E}{CR}e^{\left(-t/_{CR}\right)}$$

$\frac{dv}{dt}$= is the rate of increase of v, so minus sign shows that v is in fact **decreasing**.

In general, if $y = e^{f(x)}$ =

Then $\frac{dy}{dx} = f'(x)e^{f(x)}$

Or if $y = Ae^{f(x)}$ =

Then $\frac{dy}{dx} = Af'(x)e^{f(x)}$

So, for example

$$y = 5e^{(x^2+3x-7)}$$

$$\frac{dy}{dx}\ 5(2x+3)e^{(x^3+3x-7)}$$

You will probably find that by this means or by other means, you will be able to do these differentiations quite quickly, with practice. But don't forget the fundamentals, which you can always fall back on when in doubt:

$y = e^x \qquad \frac{dy}{dx} = e^x$

$y = Ae^x \qquad \frac{dy}{dx} = Ae^x$

And for more complicated exponentials, use

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Below is an illustration of how logs and exponentials go together.

If y = ln (x)

Then $x = e^y$

And $\frac{dy}{dx} = e^y = x$

Turn both sides upside down, $\frac{dy}{dx} = \frac{1}{x}$

Previously, this argument was used in reverse, to justify saying that $\frac{d(e^x)}{dx} = e^x$. Which comes first, ln (x) or e^x? It doesn't matter, as long as the maths is consistent!

Exercise 9

1. Find expressions for $\frac{dy}{dx}$ in the following cases:

a) $y = 7.6e^x$ b) $y = e^{5x}$

c) $y = 6e^{4x}$ d) $y = 3e^{(2x+7)}$

e) $y = e^{x/9}$ f) $y = 6e^{(x/2)}$

g) $y = 4e^{(x/6)}$ h) $y = 5e^{(x^2-5x+9)}$

i) $y = 2e^{\cos(x)}$ j) $y = Ae^{-kx}$= (A and k are constants)

Check your answers and try to sort out any difficulties before continuing.

2. If $f(x) = 5e^{(0.2x)}$, find

$f(0)$, $f'(3)$

Now check your answers.

3. A freshly poured cup of coffee is at 70^{0}C. Room temperature is 20^{0}C. The temperature T of the coffee as a function of the time t in minutes is given by the equation

T = 20 + 50$e^{(-0.4t)}$ (Notice that t= 0 then T = 20 + 50 = 70).

Calculate:

a) the rate of cooling of the coffee (^{0}C per minute) when $t = 0$

b) the temperature of the coffee after 20 minutes

c) the rate of cooling of the coffee after 20 minutes.

Now check your answers.

SUMMARY

The derivative of Ae^x with respect to x is Ae^x.

The chain rule can be used for many more complicated exponentials;

For example

$$y = 6e^{(5x^2-3x+2}$$

Put $u = 5x^2 - 3x + 2$

$$y = 6e^u$$

$$\frac{dy}{du} = 6e^u$$

$$\frac{du}{dx} = 10x - 3$$

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx} = 6e^u\ 10x - 3 = 6(10x - 3)e^{(5x^2-3x+2)}$$

ANSWERS

Exercise 1

Your answers should look like this:

1. $\frac{\mathrm{d}}{\mathrm{d}x}(3x^{7}) = 21x^{6}$

2. $\frac{\mathrm{d}}{\mathrm{d}x}(2x^{4} - 3x^{3} + 5x^{2}) = 8x^{3} - 9x^{2} + 10x$

3. $\frac{\mathrm{d}}{\mathrm{d}x}(7x^{-1} - 12x^{-2}) = -7x^{-2} + 24x^{-3}$

4. $\frac{\mathrm{d}}{\mathrm{d}x}(3x^{\frac{5}{3}} + 9x^{\frac{2}{3}}\, 3x) = 5x^{\frac{2}{3}} + 6x^{\left(\frac{-1}{3}\right)}$

If your answers don't look like that you must have forgotten how to differentiate. Remind yourself before continuing!

Now return to the text.

Exercise 2

$f(x)$ = sin(x) $f'(x)$= cos(x)

1. $f(1)$ = sin (1) = 0.8415
2. $f'(1)$ = cos (1) = 0.5403
3. $f'(0.3)$ = sin (0.3) = 0.2955
4. $f'(2.2)$ = cos(2.2) = -0.5885
5. $f'(-1)$ = cos (-1) = 0.5403

If you got them wrong, have you made sure that your calculator is set to radians? **Every** angle in this pack will be in radians.

Now return to the text.

Exercise 3

$f\ (x)$ = cos (x) $f'(x)$= -sin (x)

1. $f\ (0.75)$ = cos (0.75) = 0.7317
2. $f'(x0.75)$ = -sin(0.75) = -0.6816
3. $f'(3)$= -sin (3) = - 0.1411
4. $f(-3.1)$ = cos (-3.1) = -0.9991
5. $f'(-0.2)$= -sin (-0.2) = 0.1987

As easy as pi? The last one could be a bit confusing, with two negatives. But the main problem should be, as always, that it's just one more thing to remember.

Well, in any exam you are likely to be given $\frac{\mathrm{d}}{\mathrm{d}x}$ (sin(x)) and $\frac{\mathrm{d}}{\mathrm{d}x}$ (cos (x)) on the formula sheet. But they are so basic that it helps to know them. Perhaps you can remember them this way:

> Dee sine is cos, dee sine is,
> But dee cos is sine with a minus.

Now return to the text.

Exercise 4

1. a) The answer is 6 cos (6 x) If you didn't get it that answer, follow the working through. After this one we are going to leave out the u and du. You can write them down to follow the working, if you need to. Don't just skip over it!

$$y = \sin(6x)$$

Put

$$u = 6x$$

$$y = \sin(u)$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos(u)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 6$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\ \frac{\mathrm{d}u}{\mathrm{d}x} = \cos(u)6 = 6\cos(u) = 6\cos(6x)$$

b) $y = \cos(5x)$

$$\frac{dy}{dx} = -\sin(5x)\,5 = -5\sin(5x)$$

c) $y = 3\sin(5x)$

$$\frac{dy}{dx} = 3\cos(5x)\,5 = 15\cos(5x)$$

d) $y = \sin(x^2)$

$$\frac{dy}{dx} = \sin(x^2)\,2x = 2x\sin(x^2)$$

e) $y = \cos(7x - 3)$

$$\frac{dy}{dx} = -\sin(7x - 3)\,7 = -7\sin(7x - 3)$$

f) $y = 4\cos(7x – 3)$

$$\frac{dy}{dx} = 4(-\sin(7x - 3))\,7 = -28\sin(7x - 3)$$

g) $y = \sin(x^3 – 4x^2 + 7x - 3)$

$$\frac{dy}{dx} = \cos(x^3 – 4x^2 + 7x - 3)\,(3x^2 – 8x + 7)$$

$$= (3x^2 – 8x + 7)\cos(x^3 – 4x^2 + 7x - 3)$$

h) $y = 3\cos(5x^2 – 6x)$

$$\frac{dy}{dx} = 3(-\sin(5x^2 – 6x)\,(10x - 6))$$

That answer as it stands is correct; but it would usually be tidied up as

$$\frac{dy}{dx} = -(30x – 18)\sin(5x^2 – 6x)$$

or, by extracting a factor 6 from the bracket,

$$\frac{dy}{dx} = -6(5x – 3)\sin(5x^2 – 6x)$$

or sometimes it might be written as

$6(3 – 5x)\sin(5x^2 – 6x)$

You have to be prepared to recognise all those answers as equivalent.

i) y = sin (3 cos(x))

$$\frac{dy}{dx} = \cos(3\cos(x))(-3\sin(x))$$
$$= -3\sin(x)\cos(3\cos(x))$$

j) y = $\cos^5(x)$

$$\frac{dy}{dx} = 5\cos^4(x)(-\sin(x)) = -5\cos^4(x)\sin(x)$$

k) $f(x)$ = 3 sin (2 x +1)

$f'(x)$ = 6 cos (2 x +1)

$f(0)$ = 3 sin(0+1) = 3 sin (1) = 2.5244

$f'(0)$ = 6 cos(0+1) = 6 cos (1) = 3.2418

$f'(-1)$ = 6 cos(-2+1) = 6 cos (-1) = 3.2418

$f'(0.5)$ = 6 cos (1+1) = 6 cos (2) = -2.4969

Now return to the text.

Exercise 5

First you have to write down the formulas for f(x) and f'(x).
Then you can put in the figures.

$f(x)$ = ln (x) f'(x) f'(x) = $\frac{1}{x}$

a) $f(5)$ = ln (5) = 1.6094

b) $f'(4) = \frac{1}{4}$ = 0.25

c) $f(1)$ = ln(1) = 0

d) $f'(1) = \frac{1}{1}$ = 1

e) $f'(10) = \frac{1}{10}$ = 0.1

f) $f'(0.1) = \frac{1}{0.1}$ = 10

Only the first one needs a calculator. If you had any mistakes, perhaps it was the changes from f to f' and back again that put you off. Make sure you feel confident before moving on.

Now return to the text.

Exercise 6

1. a) $y = \ln(12x) = \ln(12) + \ln(x)$

ln (12) is just a constant (2.4849 actually, but its numerical value doesn't matter). The derivative of any constant is zero.

So $\dfrac{dy}{dx} = \dfrac{1}{x}$

b) $y = \ln(ax) = \ln(a) + \ln(x)$

The question doesn't say that a is a constant: but at this stage the question makes no sense unless a is a constant, so let's assume that it is;

Then $\dfrac{dy}{dx} = \dfrac{1}{x}$

c) $y = \ln(x^3) = 3\ln(x)$

$$\frac{dy}{dx} = 3\frac{1}{x} = \frac{3}{x}$$

If you are having any problems, read through the text again, and if necessary talk to your tutor.

d) $y = \ln(3x) = \ln(3) + \ln(x^2) = \ln(3) + \ln(x)$

$$\frac{dy}{dx} = 2\frac{1}{x} = \frac{2}{x}$$

e) $y = \ln\left[\dfrac{1}{x^3}\right] = \ln\left(x^{-3}\right) = -3\ln(x)$

$$\frac{dy}{dx} = -3\frac{1}{x} = -\frac{3}{x}$$

f) $y = \ln(x^{\frac{1}{3}}) = \dfrac{2}{3}\ln(x)$

$$\frac{dy}{dx} = \frac{2}{3}\,\frac{1}{x} = \frac{2}{3x}$$

2. $y = \ln(3x^{-2}) = \ln(3) + \ln(x^2) = \ln(3) + 2\ln(x)$

$$\frac{dy}{dx} = 2\frac{1}{x} = \frac{2}{x}$$

Any of the expressions for y can be used to find y when x is 5; but it might be simplest to use the first:

$y = \ln(3x5^2) = \ln(75) = 4.3175$

And when x is 5, $\frac{dy}{dx} = \frac{2}{5}$ = 0.4

3. y = ln (2 x^{-3}) = ln (2) + ln(x^{-3}) = ln(2) -3 ln (x)

$\frac{dy}{dx}$ = $-3\frac{1}{x}$ = $-\frac{3}{x}$

a) There are various ways of getting this answer. I shall use the third expression:

$f(2)$ = ln (2) – 3ln (2) = -2ln (2) = - 1.3863

b) $f'(2) = -\frac{3}{2}$ = - 1.5

c) $f(0.5)$ = ln (2) – 3ln(0.5) = 2.7726

d) $f'(0.1) = -\frac{3}{0.1}$ = -30

Now return to the text.

Exercise 7

1. a) y = ln (x -3)

This one will be written out in full. If you are getting these questions right, you don't really need the full explanation, so we shall leave out u and du in the rest. You can do the same if you feel confident; just put down the working that you need – don't put down working for the sake of it. But don't rely on guesswork if you lose your way. Go back to the basic operations.

Put u = $x - 3$

y = ln (u)

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dx} = 1$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\,1 = \frac{1}{(x-3)}$$

b) y = ln (x +12)

$$\frac{dy}{dx} = \frac{1}{(x+12)}$$

c) y = ln (5 x -9)

$$\frac{dy}{dx} = \frac{1}{(5x-9)}\ 5 = \frac{5}{(5x-9)}$$

d) y = ln (x^2 + 8)

$$\frac{dy}{dx} = \frac{1}{(x^2+8)}\ 2x = \frac{2x}{(x^2+8)}$$

e) y = ln (x^2+7 x)

$$\frac{dy}{dx} = \frac{1}{(x^2+7x)}\ (2x+7) = \frac{(2x+7)}{(x^2+7x)}$$

It would also be possible to say:

y = ln (x^2 + 7 x) = ln (x (x +7))= ln (x)+ln(x +7)

Then $\frac{dy}{dx} = \frac{1}{x} + \frac{1}{(x+7)}$

This comes to the same as before. There is nothing to choose between the two answers.

f) y = ln (x^2 – 3 x +4)

$$\frac{dy}{dx} = \frac{1}{(x^2-3x+4)}\ (2x-3) = \frac{(2x-3)}{(x^2-3x+4)}$$

g) y = ln (5 x^3 -9)

$$\frac{dy}{dx} = \frac{1}{(5x^3-9)}\ 15x^2 = \frac{15x^2}{(5x^3-9)}$$

h) y = ln (cos(2 x))

$$\frac{dy}{dx} = \frac{1}{\cos(2x)}\ (-2\sin(2x)) = \frac{-2\sin(2x)}{\cos(2x)} = 2\tan(2x)$$

2. As always, differentiate first, so that you have the expressions for f(x) and f'(x). Then put in the values.

a) $f\ (x)$ = ln (2 x +1)

$$f'\ (x)\ = \frac{1}{(2x+1)}\ 2 = \frac{2}{(2x+1)}$$

$f\ (1)$ = ln(2+1) = ln(3) = 1.0986

$$f'(2) = \frac{2}{(4+1)} = \frac{2}{5} = 0.4$$

b) $f\ (x)$= ln (x– 5 x +3)

$$f'\ (x) = \frac{1}{x^2-5x+3}\ (2\,x\ \text{-}5) = \frac{(2x-5}{(x^2-5x+3)}$$

$f\ (5)$ = ln (5^2 – 5x5+3) = ln(3) = 1.0986

$$f'\ (8) = \frac{2x8-5}{(8^2-5x8+3)} = \frac{11}{27} = 0.4074$$

3. $h = H\ln\left(\frac{G}{p}\right)$ = H(ln(G) – ln(p) =Hln(G) –Hln(p)

(Focus your attention on the variables p and h. H and G are constants – just numbers. Don't let them put you off!)

a) $$\frac{dh}{dp} = -H\frac{1}{p} = -\frac{H}{p} = -\frac{7000}{p}$$

b) When p = G = 760 then $\frac{dh}{dp} = -\frac{7000}{760} = -9.21$

So for a fall in pressure of 1mm the height must increase by 9.21m.

c) When p = 76 then h = 7000 ln $\left(\frac{760}{76}\right)$

= 7000 ln (10)

= 16118m(or 16.118km)

d) When p = 76 then $\frac{dh}{dp} = -\frac{7000}{76} = -92.11$

So for a fall in pressure of 1mm the height must increase by 92.11m.

Now return to the text.

Exercise 8

$f\ (x) = e^x$ $\qquad\qquad f'\ (x) = e^x$

(This procedure should be very familiar)

a) $f\ (0) = e^0 = 1$

b) $f\ (0) = e^0 = 1$

c) $f\ (1.5) = e^{1.5} = 4.4817$

d) $f\ (1.5)\ = e^{1.5} = 4.4817$

Now return to the text.

Exercise 9

1. a) $y = 7.6\ e^x$

b) $y = e^{5x}$

u and du have been used in this one, you follow the others. Once again, you can do the same - just write down the working you need. But don't try to be too clever. If in doubt, write it out!

Put $u = 5x \quad \frac{dy}{dx} = 7.6e^x$

$y = \ e^u$

$\frac{dy}{dx} = \ e^u$

$\frac{dy}{dx} = 5$

$\frac{dy}{dx} = \ \frac{dy}{du}\frac{du}{dx} = e^u 5 = 5e^u = 5e^{5x}$

c) $y = 6e^{4x}$

$\frac{dy}{dx} = 6e^{4x}4 = 24e^{4x}$

d) $y = 3e^{(2x+7)}$

$\frac{dy}{dx} = 3e^{(2x+7)}2 = 6e^{(2x+7)}$

e) $y = e^{x}/_{9}$

$\frac{dy}{dx} = \ e^{x}/_{9}\frac{1}{9} = \frac{1}{9}e^{x}/_{9} or\ \frac{e^{x}/_{9}}{9}$

f) $y = 6e^{x/2}$

$\frac{dy}{dx} = 6e^{x/2}\frac{1}{2} = 3e^{x/2}$

g) $y = 4e^{(-x/6)}$

$\frac{dy}{dx} = 4e^{(-x/6)}\left(-\frac{1}{6}\right) = -\frac{4}{6}e^{(-x/6)} = -\frac{2}{3}e^{(-x/6)}$

Or $-\frac{2e^{(-x/6)}}{3}$

h) $y = 5e^{(x^2-5x+9)}$

$\frac{dy}{dx} = 5e^{(x^2-5x+9)}(2x-5) = 5(2x-5)e^{(x^2-5x+9)}$

i) $y = 2e^{\cos(x)}$

$\frac{dy}{dx} = 2e^{\cos(x)}(-\sin(x)) = -2\sin(x)e^{\cos(x)}$

j) $y = Ae^{-kx}$

$\frac{dy}{dx} = Ae^{-kx}(-k) = -Ake^{-kx}$

2. $f\,(x) = 5e^{(0.2x)}$

$f'^{(x)} = 5e^{(0.2x)}0.2 = e^{(0.2x)}$

$f\,(0) = 5e^0 = 5 \quad f'(0) = e^0 = 1$

$f'^{(3)} = e^{(0.2\times 3)} = e^{0.6}$= 1.8221

3. $T = 20 + 50e^{(-0.04t)}$

(This type of equation arises from Newton's law of cooling. Isaac had lots of ideas, and most of them were good)

This question is just another case of getting the formulas (for T and $\frac{dT}{dt}$) and putting in the values.

The fact that it's related to practical situation doesn't change that.

Differentiate : $\frac{dT}{dt} = 50e^{(-0.04t)}\ (-0.04) = -2^{(-0.04t)}$

a) When $t = 0$ then $\frac{dT}{dt} = -2e^0 = -2$

So the rate of cooling at first is 2^0C per minute.

b) When $t = 20$ then $T = 20 + 50e^{(-0.04\times 20)}$

$$
\begin{aligned}
&= 20 + 50e^{(-0.8)} \\
&= 20 + 22.47 \\
&= 42.47
\end{aligned}
$$

So the temperature after 20 minutes is 42.47^{0}C.

c) When $t = 20$ then $\frac{dT}{dt} = -2e^{(-0.8)} = -0.90$

So after 20 minutes the rate of cooling is 0.90^{0}C per minute (correct to two decimal places).

Now return to the text.