MATRICES

A matrix is a rectangular display, or array, of numbers arranged in rows (horizontal) and columns (vertical) which is used to present information. It is enclosed by a large pair of brackets.
The plural of matrix is matrices.
Arrays of numbers have been used to present information, or could have been used to present information in previous packs.

The table below is an array of numbers:

Years	5%	6%	7%
1	1.050	1.060	1.070
2	1.103	1.124	1.145
3	1.158	1.191	1.225
4	1.216	1.262	1.311
5	1.276	1.338	1.403

row

column

If this table were being used by those who understand its purpose the row and column headings

Years	5%	6%	7%
1			
2			
3			
4			
5			

could be omitted and the number array presented as a matrix with five rows and three columns.

$$\begin{array}{r} \\ \\ \\ \text{fourth row} \rightarrow \\ \\ \end{array} \begin{pmatrix} 1.050 & 1.060 & 1.070 \\ 1.103 & 1.124 & 1.145 \\ 1.158 & 1.191 & 1.403 \\ 1.225 & 1.216 & 1.262 \\ 1.311 & 1.276 & 1.338 \end{pmatrix}$$

The matrix is said to be of order five by three. The curved brackets around a matrix hold it together and show that it is complete. In reading the matrix the number on the fourth row, in the second column is 1.262 and this is the amount which £1 would increase to if it had been accumulating interest at 6% for four years.

Men	Hours	Days
12	8	5
10	6	?

If the table is completed and the column headings omitted the information becomes a matrix with two rows and three columns:

$$\begin{matrix} 12 & 8 & 5 \\ 10 & 6 & ? \end{matrix}$$

This matrix is of order two by three, 2 x 3.

Very often, before a chart is drawn up, the data collected is tabulated. The array of numbers below shows the numbers of children wearing different coloured shirts.

Colour	Boys	Girls	Total
Red	3	12	15
Green	2	3	5
Blue	4	1	5
Black	2	1	3
White	0	2	2

If the row and column headings are omitted, the array of data is presented as a matrix of order five by three,

$$\begin{matrix} 3 & 12 & 15 \\ 2 & 3 & 5 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \\ 0 & 2 & 2 \end{matrix}$$

The data has been collected from only one class of students. If the shirt colours of another class were noted and presented in the same way, the matrix might be:

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \\ 7 & 0 & 7 \\ 9 & 5 & 14 \end{matrix}$$

Both these matrices present data about a class of students and the columns of shirts they wear. Both matrices have five rows and three columns. In the second class nobody was wearing a red shirt. The first row of the matrix is (0 0 0). If this row had been left out the second matrix would only have four rows and it would not be possible to compare it directly with the matrix for the first class.

$$\begin{matrix} 0 & 1 & 1 \\ 0 & 5 & 5 \\ 7 & 0 & 7 \\ 9 & 5 & 14 \end{matrix}$$

It would not be clear, from the matrix, which row was missing.

Including the zero, or null, row makes it possible to combine the two matrices.

3	12	15		0	0	0
2	3	5		0	1	1
4	1	5	and	0	5	5
2	1	3		7	0	7
0	2	2		9	5	14

To produce a third:

3	12	15
2	4	6
4	6	10
9	1	10
9	7	16

In these matrices, the two numbers in the first row and the first column; 3 and 0, both represent a number of boys wearing red shirts. If they are added, 3 + 0 = 3, the total is the number of students wearing red shirts in the two classes.

Similarly, the two numbers in the fifth row and the third column, 2 and 14, both represent a number of students wearing white shirts. If they are added, 2 + 14 = 16, the total is the number of students wearing white shirts in the two classes. These totals can be written in a third matrix, giving information about the two classes combined.

MATRIX ADDITION

Matrices can be added:

1. If they are of the **same order**.
   (A 5 x 3 matrix can only be added to a 5 x 3 matrix. If **cannot** be added to a 4 x 3 matrix) and;
2. If it **makes sense to add them**.

However it would make no sense to add together a matrix showing the growth of capital accumulating compound interest and one showing the clothes worn by students. The resulting matrix would have no meaning.

To add matrices, corresponding numbers, or elements, in each matrix are added.

3	12	15		0	0	0		3+0	12+0	15+0		3	12	15
2	3	5		0	1	1		2+0	3+1	5+1		2	4	6
4	1	5	+	0	5	5	=	4+0	1+5	5+5	=	4	6	10
2	1	3		7	0	0		2+7	1+0	3+7		9	1	10
0	2	2		9	5	14		0+9	2+5	2+14		9	7	16

Each number, or element, of the new matrix has a meaning. The element on the fourth row in the second column is the number of girls wearing black shirts in the two classes.

SUBTRACTION OF MATRICES

We can also subtract matrices. Once again matrices can **only** be subtracted:
1. If they are of the same order; and
2. If it makes sense to subtract them.

To subtract matrices, the corresponding numbers, or elements, in each matrix are subtracted.

Example

$$\begin{array}{cccccccccccc} 6 & 4 & & 3 & 0 & & 6-3 & 4-0 & & 3 & 4 \\ 2 & 1 & - & 7 & 5 & = & 2-7 & 1-5 & = & -5 & -4 \\ 3 & 2 & & 1 & -1 & & 3-1 & 2- -1 & & 2 & 3 \end{array}$$

NB - - = +

MULTIPLICATION OF MATRICES BY "NORMAL" NUMBERS

It may also be sensible to multiply a matrix by a normal number. e.g. Three children buy sweets each **week** as shown by the following matrix

	Mars Bar	Snickers	Bounty
Peter	3	2	4
Joanne	1	5	0
Mark	4	1	4

To calculate the quantities of sweets bought during a year we multiply each number in the matrix by 52 (52 weeks = 1 year)

$$\begin{array}{ccccccccc} & & 3 & 2 & 4 & & 156 & 104 & 208 \\ 52 \times & 1 & 5 & 0 & = & & 52 & 260 & 0 \\ & & 4 & 1 & 4 & & 208 & 52 & 208 \end{array}$$

Sometimes we use letters to represent matrices e.g:

$$A = \begin{matrix} 2 & 4 \\ 1 & 6 \end{matrix} \qquad B = \begin{matrix} 0 & 3 \\ 2 & 5 \end{matrix} \qquad C = \begin{matrix} 4 & -1 \\ 2 & 3 \end{matrix}$$

Calculate:- i) A + B, ii) A – C iii) 2C

i) A + B

$$\begin{matrix} 2 & 4 \\ 1 & 6 \end{matrix} + \begin{matrix} 0 & 3 \\ 2 & 5 \end{matrix} = \begin{matrix} 2+0 & 4+3 \\ 1+2 & 6+5 \end{matrix} = \begin{matrix} 2 & 7 \\ 3 & 11 \end{matrix}$$

ii) A - C

$$\begin{matrix} 2 & 4 \\ 1 & 6 \end{matrix} - \begin{matrix} 4 & -1 \\ 2 & 3 \end{matrix} = \begin{matrix} 2-4 & 4-1 \\ 1-2 & 6-3 \end{matrix} = \begin{matrix} -2 & 5 \\ -1 & 3 \end{matrix}$$

iii) 2C

$$2 \times \begin{matrix} 4 & -1 \\ 2 & 3 \end{matrix} = \begin{matrix} 2x4 & 2x-1 \\ 2x2 & 2x3 \end{matrix} = \begin{matrix} 8 & -2 \\ 4 & 6 \end{matrix}$$

Exercise 1

1. What order is the matrix $\begin{pmatrix} 4 & 2 \\ 3 & 1 \\ 0 & 1 \end{pmatrix}$?

2. This matrix shows the number of bottles of milk ordered each day for a week by 3 householders

		M	T	W	Th	F	S	S
Household	A	1	1	1	1	1	2	2
	B	2	2	3	2	2	3	3
	C	1	2	1	2	1	2	2

This matrix gives the same information for the following week:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$

a) Are the matrices of the same order?

b) What is the order of the matrices?

c) Would it make sense to add them?

d) If so give the matrix formed by adding these two together.

3. $A = \begin{pmatrix} 4 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -2 \end{pmatrix}$ $C = \begin{pmatrix} 5 & 2 \end{pmatrix}$

Calculate

a) A + C b) A + B c) 4B

MULTIPLICATION OF MATRICES

The process of multiplying one matrix by another matrix is a little more complicated. In an earlier example we looked at children buying sweets each week as described by the matrix:

	Mars Bar	Snickers	Bounty
Peter	3	2	4
Joanne	1	5	0
Mark	4	1	4

Suppose the prices for the first week in January are described as:

	Jan Price
Mars Bar	20
Snickers	21
Bounty	22

Question How much did Peter spend that week?
Answer Peter spends 3 x 20 + 2 x 21 + 4 x 22 = 190

Notice that this process involves **multiplying** pairs of numbers and then **adding**.

Similarly	Joanne spends	1 x 20 + 5 x 21 + 0 x 22 = 125
and	Mark spends	4 x 20 + 1 x 21 + 4 x 22 = 189.

The totals spent could be described as

	Total
Peter	190
Joanne	125
Mark	189

We can summarise this process by writing

$$\begin{matrix} 3 & 2 & 4 \\ 1 & 5 & 0 \\ 4 & 1 & 4 \end{matrix} \quad x \quad \begin{matrix} 20 \\ 21 \\ 22 \end{matrix} \quad = \quad \begin{matrix} 3x20 + & 2x21 + & 4x22 \\ 1x20 + & 5x21 + & 0x22 = \\ 4x20 + & 1x21 + & 4x22 \end{matrix} \quad \begin{matrix} 190 \\ 125 \\ 189 \end{matrix}$$

Note

A “multiplication” sign is used between the two matrices on the left hand side but the process involves multiplying pairs of numbers and **adding**.

Notice the pattern: the first row column space in the answer (C) is filled by combining the **first row** of A and the **first column** of B.

Using the multiply and add process.

$$\begin{matrix} 3 & 2 & 4 \end{matrix} \quad \begin{matrix} 20 \\ 21 \\ 22 \end{matrix} \quad = \quad (3x20) + \quad (2x21) + \quad (4x22) = \quad 190$$

Now suppose prices were to change and in February were given by $\begin{matrix} 25 \\ 22 \\ 19 \end{matrix}$

then the process could be repeated with the new prices.

$$\begin{matrix} 3 & 2 & 4 \\ 1 & 5 & 0 \\ 4 & 1 & 4 \end{matrix} \quad x \quad \begin{matrix} 25 \\ 22 \\ 19 \end{matrix} \quad = \quad \begin{matrix} 3x25 + & 2x22 + & 4x19 \\ 1x25 + & 5x22 + & 0x19 = \\ 4x25 + & 1x22 + & 4x19 \end{matrix} \quad \begin{matrix} 195 \\ 135 \\ 198 \end{matrix}$$

We could in fact neatly describe these two stages by writing:

$$\underset{P}{\begin{matrix} 3 & 2 & 4 \\ 1 & 5 & 0 \\ 4 & 1 & 4 \end{matrix}} \quad \underset{x}{x} \quad \underset{Q}{\begin{matrix} 20 & 25 \\ 21 & 22 \\ 22 & 19 \end{matrix}} \quad = \quad \underset{R}{\begin{matrix} 190 & 195 \\ 125 & 135 \\ 189 & 198 \end{matrix}}$$

Notice that the result in the second row, second column of the answer was obtained by combining the second row of matrix P with the second column of matrix Q.

$$\begin{matrix} . & . & . \\ 1 & 5 & 0 \\ . & . & . \end{matrix} \quad x \quad \begin{matrix} . & 25 \\ . & 22 \\ . & 19 \end{matrix} \quad = \quad \begin{matrix} . & . \\ . & 135 \\ . & . \end{matrix}$$

We see that this process requires the two matrices to **match**. The number of **columns** in the first must match the number of **rows** in the second.

Order of First matrix	Order of second matrix
3 x rows x 3 columns	3 rows x 2 columns

[The order of the answer matrix is given by outer numbers i.e. 3 x 2 in the example above].

GENERAL MATRIX MULTIPLICATION

NB. Two Matrices can only be multiplied, if the number of columns of the first is equal to the number rows of the second. If this is so, the two matrices are said to be conformable for multiplication.

Thus the following matrices would be conformable for multiplication.

$$\begin{pmatrix} 2 & -1 \\ 4 & -3 \\ 0 & -1 \end{pmatrix} \quad x \quad \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$$

(3 x②) ⟵ same ⟶ (②x 3) Resulting matrix would be (2 x 3)

$$\begin{pmatrix} 3 & 1 & -2 & 6 \\ 2 & -1 & 8 & 4 \end{pmatrix} \quad x \quad \begin{pmatrix} 6 & 5 & 4 \\ 1 & -3 & 0 \\ -1 & 2 & 7 \\ 8 & 4 & 9 \end{pmatrix}$$

(2 x ④) ⟵ same ⟶ (④x 3) Resulting matrix would be (2 x 3)

$$\begin{pmatrix} 6 & 8 & 4 \\ -2 & 3 & 0 \end{pmatrix} \quad x \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(2 x ③) ⟵ same ⟶ (③ x 1) Resulting matrix would be (2 x 1)

The following matrices are **not** conformable for multiplication.

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix} \quad x \quad \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 4 & 5 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} \quad x \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 9 & 7 & 5 \end{pmatrix}$$

(2 x③) ⟷ (④x 2) (3 x ②) ⟷ (③x 3)

not the same not the same

$$\begin{pmatrix} -2 \\ 6 \\ 4 \\ 5 \end{pmatrix} \quad x \quad \begin{pmatrix} 4 & 5 & 1 & 0 \\ 6 & -3 & 4 & 8 \end{pmatrix}$$

(4 x 1) $\longleftrightarrow$ (2 x 4)

not the same

some further examples of matrix multiplication are shown below:

Example 1

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad x \quad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1x5+2x7 & 1x6+2x8 \\ 3x5+4x7 & 3x6+4x8 \end{pmatrix}$$

(2 x 2) $\longleftarrow$ same $\longrightarrow$ (2 x 2)

$$= \begin{pmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{pmatrix}$$

$\therefore$ Multiplication can take place $= \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$ a <u>(2 x 2)</u> matrix

Product of 1st row of A and 1st column of B

Product of 1st row of A and 2nd column of B

(A) (B)

$$\therefore \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Product of 2nd row of A and 1st column of B

Product of 2nd row of A and 2nd column of B

Example 2

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad x \quad \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1x1+2x3+3x2 & 1x0+2x2+3x4 \\ 4x1+5x3+6x2 & 4x0+5x2+6x4 \end{pmatrix}$$

(2 x 3) (3 x 2) $= \begin{pmatrix} 1+6+6 & 0+4+12 \\ 4+15+12 & 0+10+24 \end{pmatrix} = \begin{pmatrix} 13 & 16 \\ 31 & 24 \end{pmatrix}$

same so multiplication can take place <u>(2 x 2)</u>

Example 3

$$\begin{pmatrix} 4 & 1 & 0 \\ 5 & 2 & 7 \end{pmatrix} \text{ x } \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4\text{x}5 + 1\text{x}4 + 0\text{x}1 \\ 5\text{x}5 + 2\text{x}4 + 7\text{x}1 \end{pmatrix}$$

(2 x 3) (3 x 1)

$$= \begin{pmatrix} 20 + 4 + 0 \\ 25 + 8 + 7 \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ 40 \end{pmatrix}$$ a (2 x 1) matrix

Example 4

$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}$, form the matrix products

a) AB   b) BA

a) $$AB = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \text{ x } \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 2\text{x}4 + 3\text{x}3 & 2\text{x}1 + 3\text{x}(-2) \\ 1\text{x}4 + 4\text{x}3 & 1\text{x}1 + 4\text{x}(-2) \end{pmatrix}$$

(2 x 2) (2 x 2)

$$= \begin{pmatrix} 8 + 9 & 2 - 6 \\ 4 + 12 & 1 - 8 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & -4 \\ 16 & -7 \end{pmatrix}$$ a (2 x 2) matrix

b) $$BA = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix} \text{ x } \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4\text{x}2 + 1\text{x}1 & 4\text{x}3 + 1\text{x}4 \\ 3\text{x}2 + (-2)\text{x}1 & 3\text{x}3 + (-2)\text{x}4 \end{pmatrix} = \begin{pmatrix} 8 + 1 & 12 + 4 \\ 6 - 2 & 9 - 8 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 16 \\ 4 & 1 \end{pmatrix}$$ a (2 x 2) matrix

Notice here that the order of multiplication is important as the result of AB is not the same as for BA.

Example 5

$(3 \quad 2) \quad \times \quad \begin{pmatrix} 5 & 4 \\ 1 & 0 \end{pmatrix} = (3x5 + 2x1 \quad 3x4x2x0)$

(1 x 2) ← same → (2 x 2) = (17 12)

Example 6

$\begin{pmatrix} 2 & 1 \\ 6 & 7 \end{pmatrix} \quad \times \quad (3 \quad 2)$

(2 x 2) ←→ (1 x 2)

not the same \ IMPOSSIBLE

Notes

1. Even if two matrices A and B are conformable for multiplication, in general $AB \neq BA$

2. Two matrices of orders (m x n) and (p x q) are conformable for multiplication if number columns in first = number of rows in second i.e. if n=p. The order of the resulting matrix will be (m x q).

Exercise 2

Multiply out the following matrices

1. $\begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix}$

3. $\begin{pmatrix} 6 & 7 \\ 1 & 2 \\ 10 & 5 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

4. $\begin{pmatrix} 5 & 6 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

5. $\begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 5 & 7 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

7. $\begin{pmatrix} 5 & 3 \\ 7 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

What do you notice? Does anything special happen?

8. $\begin{pmatrix} 5 & 6 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

9. $\begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 5 & 6 \\ 1 & 2 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

THE IDENTITY FOR MULTIPLICATION. A SPECIAL MATRIX

This is defined as the matrix which causes no change when used in multiplication.

$$\begin{pmatrix} 4 & 6 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 5 \end{pmatrix}$$

The answer to this is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(Look at questions 7 and 10 in previous exercise)

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a special type of 2 x 2 matrix and is called the Identity Matrix for multiplication

of all **2 x 2 matrices**. It is usually denoted by the letter I (I for identity – get it!) and it always works! **REMEMBER IT!**

ANSWERS

Exercise 1

1. 3×2

2. a) yes

   b) 3×7

   c) yes

   d) $\begin{matrix} 2 \\ 4 \\ 2 \end{matrix} \quad \begin{matrix} 2 \\ 4 \\ 3 \end{matrix} \quad \begin{matrix} 3 \\ 6 \\ 2 \end{matrix} \quad \begin{matrix} 2 \\ 4 \\ 4 \end{matrix} \quad \begin{matrix} 2 \\ 5 \\ 2 \end{matrix} \quad \begin{matrix} 4 \\ 6 \\ 4 \end{matrix} \quad \begin{matrix} 4 \\ 6 \\ 4 \end{matrix}$

3. a) $\begin{matrix} 9 & 2 \\ 3 & 6 \end{matrix}$

   b) $\begin{matrix} 6 & -2 \\ 10 & 6 \end{matrix}$

   c) $\begin{matrix} 8 & -8 \\ 16 & 12 \end{matrix}$

Exercise 2

Matrix Multiplication

1. IMPOSSIBLE

2. $\begin{pmatrix} 23 & 24 \\ 37 & 42 \end{pmatrix}$

3. $\begin{pmatrix} 18 \\ 3 \\ 30 \end{pmatrix}$

4. (-4)

5. $\begin{pmatrix} 15 & 0 \\ 11 & 7 \end{pmatrix}$

6. $\begin{pmatrix} 13 & 16 \\ 11 & 14 \end{pmatrix}$

7. $\begin{pmatrix} 5 & 3 \\ 7 & 4 \end{pmatrix}$

8. $\begin{pmatrix} 15 \\ 3 \end{pmatrix}$

9. IMPOSSIBLE

10. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$