Comparing Geometries of Reflection Groups

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Abstract

Coxeter groups give us a way of describing groups with respect to a presentation, however, the same group may have multiple descriptions. In this project we look to address this by considering how these presentations look when converted to a geometric model – namely Cayley graphs.

We consider the standard 2-generator Coxeter presentation of dihedral groups and identify trends in their Cayley graphs. We then discover some issues in 3-generator Coxeter systems of dihedral groups and provide some criteria to ensure these systems can indeed exist. We conclude the paper by producing the Cayley graphs of these 3-generator Coxeter systems and contrast them with the previous isomorphic models.

Keywords: *Coxeter Systems, Dihedral Groups, Cayley Graphs, Pseudotranspositions, Blow-ups*

Introduction

This project was very generously funded by the Undergraduate Research Opportunities Scheme (UROS); a competitive bursary scheme that allows for undergraduates to produce research papers. This scheme embodies the key principles of the University of Lincoln – 'Student as Producer' – and allows students the opportunity to gain an understanding of how modern research is conducted, a feat often considered out of reach prior to doctoral level.

One branch of modern research in group theory aims to identify 3 key traits of finitely presented infinite groups. Infinite groups have an infinite number of group elements so trying to understand them all is a hopeless task. Proposed by Max Dehn in the 1940s, the following problems provide clarity on infinite groups:

- The Isomorphism Problem – In how many ways can we express the same group structure?
- The Conjugacy Problem – Is there a way to tell whether groups elements are conjugate in a given group?
- The Word Problem – How many words represent the same group element?

Coxeter groups form an important (infinite) family of groups that are, by construction, given by a special type of presentation – they are constructed using elements that are their own inverse. These groups encode the concept of mirror symmetries in the three model spaces, namely affine, spherical, and hyperbolic spaces. The problem is: how to tell whether a Coxeter group admits a single special presentation?

Preface: When we refer to D_n we refer to the dihedral group of a $n-$gon.

Project Background

Given a group G, we have a group presentation that description of our elements and how they are related. To form a group presentation, we must first identify the generators, a set of elements denoted S such that every element of the group can be expressed as a product of powers of these generators. Then we identify the set of relators, a set of products of powers of generators that equal to the identity element which we denote R. We are then able to take S and R and form our group presentation:

$$G = \langle S | R \rangle$$

Group presentations are not unique, and a single group may have several different presentations - this is where the Isomorphism Problem arises from.

Coxeter groups, also referred to as reflection groups, are groups with presentations which follow a special set of rules. Given a generating set S, we can identify the relators, but for Coxeter groups we usually format them into a matrix for ease.

Definition 1 [6]: A Coxeter matrix M_S is a $n \times n$ symmetric matrix with entries $M_{i,j} \in M_S$

$$M_{i,j} = \begin{cases} 1, & if\ i = j, \\ m \in \mathbb{N} \cup \{\infty\}, & if\ i \neq j. \end{cases}$$

Matrix entries $M_{i,j} = 1$ when $i = j$ tells us that each our generators are order 2, that is, they are their own inverses – like how reflecting an image on an axis is undone by reflecting it again across the same axis (this notion is where they get the name reflection groups). If $M_{i,j} = \infty$ then we can conclude there is no relationship between this pair of generators.

Definition 2 [6]: Let M_S be a Coxeter matrix of size $n \times n$, and $S = \{s_1, s_2, \dots\}$. The group W, defined by the finite presentation

$$W = \left\langle S \middle| \left(s_i s_j\right)^{M_{i,j}}, i,j \in \{1, \dots, n\} \right\rangle,$$

is a Coxeter group attached to a Coxeter system (W, S, M_S).

These groups encode symmetries on one of the three model spaces: Hyperbolic, Affine, or Spherical. Groups acting on a spherical space are the only groups that are finite.

If we are working with a special type of Coxeter groups called triangle groups, named after them having 3 generators, we can determine the space on which they are acting rather quickly:

Theorem 1 [1]: A triangle group $\Delta(a,b,c) = \langle s,t,u | s^2, t^2, u^2, (st)^a, (su)^b, (tu)^c \rangle$ is

(i) Spherical $\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$,

(ii) Affine $\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$,

(iii) Hyperbolic $\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$.

Furthermore, we able to visualise groups by creating Cayley graphs. Cayley graphs tell us elements interact with respect to the generating set. More formally

Definition 3 [3]: For a group G with a symmetric generating set S, the Cayley graph, denoted $\Gamma(G,S)$ or $Cay(G,S)$, is defined as:
The set of elements of G is the vertex set, and 2 vertices g, h are connected by an edge if there exists some $s \in S$ such that $g = sh$.

Literature Review

As previously mentioned, groups do not have a unique group presentation- there exists multiple ways to express the same group with varying sizes of generating sets and relators.

To keep within the length of the UROS paper, we are restricting ourselves to considering only dihedral groups, an important subfamily of Coxeter groups and an important family in group theory more generally. We assume some familiarity with dihedral groups; however, we refer to reader to [2] for a refresher if necessary.

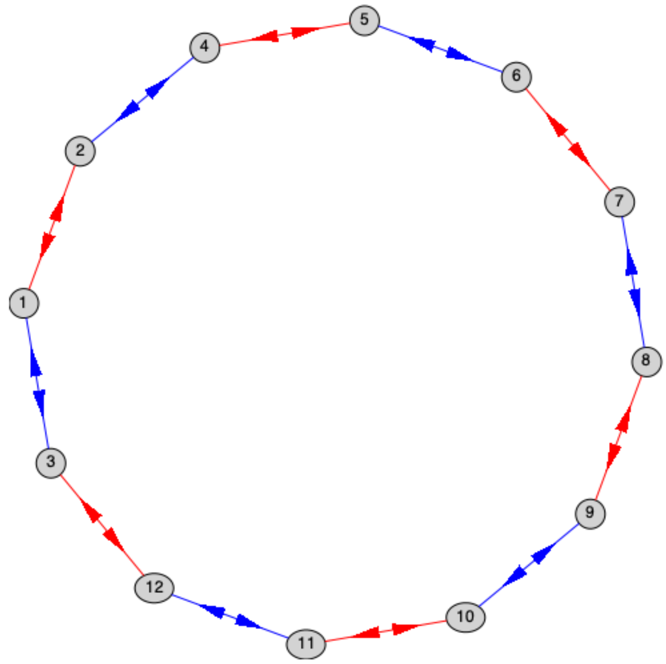
We can generalise the generating sets for dihedral groups with 2-generators. The standard presentation of D_n is $\langle s,t | s^2, t^2, (st)^n \rangle$. We can also provide more clarity on what s and t represent:

Lemma 1: For a Coxeter system $J = \langle s,t | s^2, t^2, (st)^n \rangle$ which is isomorphic to the dihedral group D_{2n}, the elements s and t are reflections that differ by $\pm\frac{\pi}{n}$.

Now that we have established how to form a dihedral group with 2 reflections using a Coxeter presentation, we can produce our first Cayley graph:

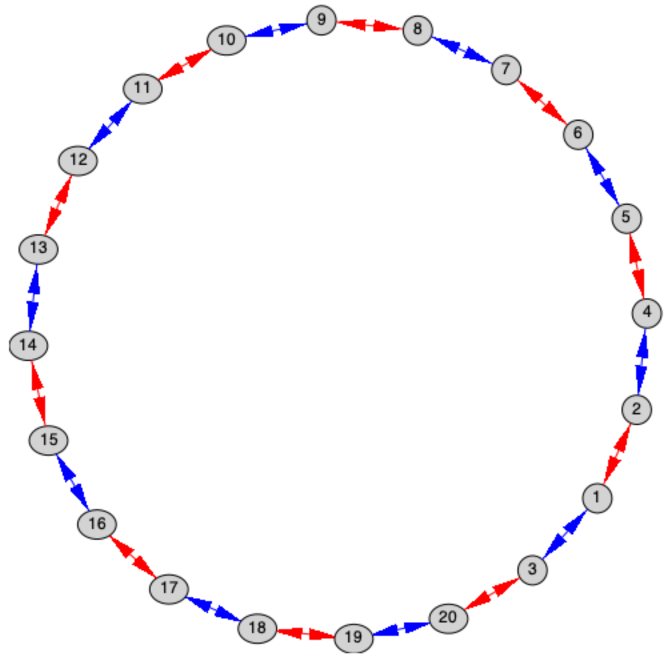
Example 1: Take D_6 with the generating set $S = \{s,t\}$ and the presentation $W = \langle s,t | s^2, t^2, (st)^6 \rangle$. We draw the Cayley graph $\Gamma(W,S)$.

We can note that this Cayley graph produces a dodecagon.

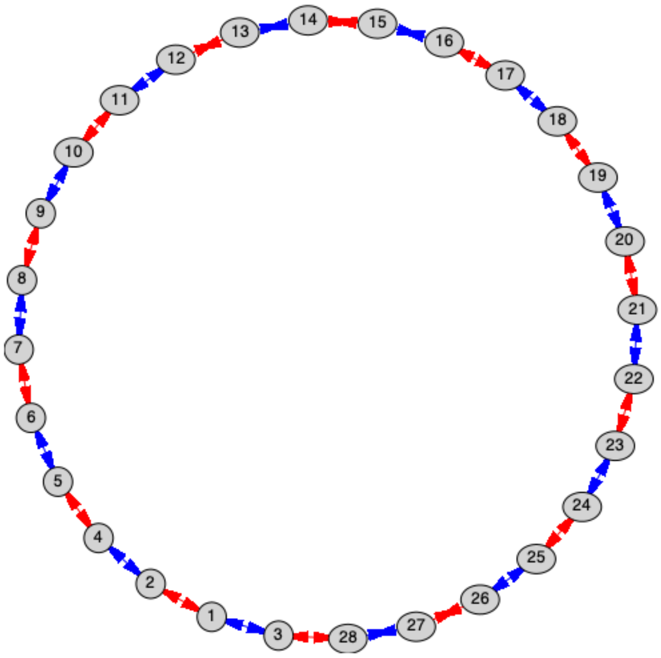


D_n forming a $2n$-gon with the 2-generator Coxeter presentation is a trend that continues, as seen in the following examples:

D_{10} with the generating set $S = \{s, t\}$ and the presentation $W = \langle s, t | s^2, t^2, (st)^{10} \rangle$ produces a Cayley graph that forms an icosagon (20-gon).



D_{14} with the generating set $S = \{s, t\}$ and the presentation $W = \langle s, t | s^2, t^2, (st)^{10} \rangle$ produces a Cayley graph that forms an icosikaioctagon (28-gon).



Suppose we wanted to produce a 3-generator Coxeter group that is isomorphic to a finite dihedral group, we would need to find a generating set of reflections that act on a spherical space. Here we encounter our first obstruction: the majority of generating sets produce act on the hyperbolic space.

Proposition 1: Let $s, t, u \in D_n$ be any three reflections in a dihedral group with $n > 4$, and order $2n$. Let $a = M_{s,t}, b = M_{s,u},$ and $c = M_{t,u}$ be the orders of $st, su, tu \in D_n$ respectively, then the triangle group $\Delta(a, b, c)$ is hyperbolic if, and only if, the greatest angle between the hyperplanes of reflections is less than $\frac{\pi}{2}$.

Proof: Fix 3 reflections in D_n and label anticlockwise s, t, u such that the angle between s and u is $\alpha_{s,u} = \alpha_{s,t} + \alpha_{t,u}$ where $\alpha_{x,y}$ is the smallest angle between the lines of reflection of x and y; the greatest any angle $\alpha_{x,y}$ may be is $\frac{\pi}{2}$ (since if $\alpha_{x,y} > \frac{\pi}{2}$, we would take the supplementary angle).

The smallest possible angle between any 2 reflections may be recognised as the angle between 2 adjacent reflections, and this angle is exactly $\frac{\pi}{n}$.

Furthermore, the angle between two reflections can be expressed as a multiple of this smallest angle: that is for any $x, y \in \{s, t, u\}$, $\alpha_{x,y} = \frac{\gamma\pi}{2n}$ where $\gamma \in \mathbb{N}$ and $\gamma < 2n$.

Turning to [6], we may also define this angle as:

$$\alpha_{x,y} = \frac{\pi}{M_{x,y}}.$$

By rearranging and substituting our previous definition of $\alpha_{x,y}$, we obtain:

$$M_{x,y} = \frac{2n}{\gamma}.$$

Recalling *Theorem 1*, we want to verify

$$\frac{1}{M_{s,t}} + \frac{1}{M_{s,u}} + \frac{1}{M_{t,u}} < 1.$$

Substituting our derived values above gives:

$$\frac{1}{\frac{2n}{\gamma}} + \frac{1}{\frac{2n}{\delta}} + \frac{1}{\frac{2n}{\epsilon}} = \frac{\gamma}{2n} + \frac{\delta}{2n} + \frac{\epsilon}{2n} = \frac{\gamma + \delta + \epsilon}{2n}.$$

Since $\alpha_{s,u} = \alpha_{s,t} + \alpha_{t,u}$, we can express$\alpha_{s,u} = \frac{(\gamma+\delta)\pi}{2n}$. this allows us to replace ϵ in our working with $\gamma + \delta$.Returning to our working:

$$\begin{aligned} \frac{\gamma + \delta + \epsilon}{2n} &= \frac{\gamma + \delta + \gamma + \delta}{2n} \\ &= \frac{2(\gamma + \delta)}{2n} \\ &= \frac{\gamma + \delta}{n}. \end{aligned}$$

Recalling that $\alpha_{s,u} = \frac{(\gamma+\delta)\pi}{2n}$, we observe that

$$\frac{1}{M_{s,t}} + \frac{1}{M_{s,u}} + \frac{1}{M_{t,u}} < 1$$

If, and only if, $\alpha_{s,u} = \frac{(\gamma+\delta)\pi}{2n} < \frac{\pi}{2}$. ■

From this proof, we obtain an important corollary:

Corollary 1: A mapping $f := \Delta(M_{s,t}, M_{s,u}, M_{t,u}) \to D_n$ such that $f(a) = s, f(b) = t, f(c) = u$ is never an isomorphism; that is, no set of three reflections $\{s, t, u\} \subseteq D_n$ gives rise to a Coxeter system for D_n.

Proof: Since the greatest angle between two reflections in D_n is $\frac{\pi}{2}$, the triangle group $\Delta(M_{s,t}, M_{s,u}, M_{t,u})$ is never spherical, thus is not finite and isomorphic to D_n. ■

Methodology

As stated, the same group may have multiple presentations. A dihedral group can be generated by any 2 reflections provided they differ by $\frac{\pi}{n}$. We can use these generators to produce another presentation but with 3 generators – we can achieve this using blow-ups. But first we must define a few terms:

Definition 4 [1]: The longest element of a Coxeter group is the unique element of maximal length in a finite Coxeter group with respect to the chosen generating set consisting of simple reflections.

In a 2-generator Coxeter group $W = \langle s, t | s^2, t^2, (st)^n \rangle$, this element is equal to $(st)^{\frac{n}{2}}$ (and since the Coxeter matrix is symmetric, it is also equal to $(ts)^{\frac{n}{2}}$).

Definition 5 [5]: A generator of a Coxeter group t is a pseudotransposition if given a Coxeter system (S, W, M_S), the following conditions are met:

1. t is contained in a subset $J \subseteq S$ where for any $s \in S \backslash J$,either $M_{s,t} = \infty$ or $M_{s,u} = 2$ for all $u \in J$.
2. The parabolic subgroup generated by J,denoted W_J, is either of type B_k where > 2, or $I_2(2k)$ for $k > 2$ where k is odd.
3. In the case of type B_k, $M_{t,u} = 2$ for all $u \in J$ except for one element $v \in J$, for which $M_{t,v} = 4$.

Since we are only considering dihedral groups, we will be concerned with the instances where $M_{s,u} = 2$ for all $u \in J$ and W_J is of type $I_2(2k)$.

Pseudotranspositions allow us to transform a Coxeter group with n generators to an isomorphic group with $n + 1$ generators whilst maintaining the underlying structure. This can be achieved using blowups.

Definition 6 [5]: Given a Coxeter system (S, W, M_S) for which a pseudotransposition has been identified, we are able to define a new generating set:

$$S' = \{tvt, w_J\} \cup (S \backslash \{t\})$$

Where t is our pseudotransposition as we have seen in *Definition 5*, $v \in J$ such that $M_{t,v} \neq 2$, and w_J is the element corresponding to the longest element in W_J.
The new Coxeter matrix $M_{S'}$ has entries

$$M'_{x,y} = \begin{cases} \frac{M_{tv}}{2} & if\ x = tvt, y = v, \\ M_{x,y} & if\ x, y \in S \cap S', \\ M_{v,y} & if\ x = tvt\ and\ y \in J \setminus \{t, v\}, \\ 2 & if\ x = w_J\ and\ y \in (\{tvt\} \cup J) \setminus \{t\}, \\ \infty & if\ x \in \{tvt, w_J\} and\ y \in S \setminus J\ and\ M_{t,y} = \infty, \\ 2 & if\ x \in \{tvt, w_J\} and\ y \in S \setminus J\ and\ M_{t,y} \neq \infty, \end{cases}$$

The Coxeter system $(S', W, M_{S'})$ is called a blow-up of (S, W, M_S).

Strictly speaking, this definition is also a lemma due to the nontriviality of the statement hence we refer the reader to [5] for a complete proof.

Blowing up a Coxeter presentation allows us to obtain a new presentation that is isomorphic to what we started with. For example, if we take the standard presentation of D_6, we can perform a blowup:

Example 2: Starting with our original presentation

$$W = \langle s, t | s^2, t^2, (st)^6 \rangle$$

with $S = \{s, t\}$.

We must first find our pseudotransposition.

Since J must either be a subset of, or equal to, the set of generators, we can identify that we do not have any pair of generators with no torsion thus we must find a subset for which elements of J have order 2 with $u \in S \backslash J$. If we take $J = S$, then $S \backslash J = \emptyset$ meaning $M_{s,u} = 2$ since all elements of J are of order 2.

The parabolic subgroup W_J is immediate recognizable as type $I_2(6)$, and 6 is of the form $2k$ where k is odd and $k > 2$, thus s or t may provide as our pseudotransposition.

Taking t as our pseudotransposition, we may now generate an isomorphic system with 3 generators using a blow up. Since $M_{s,t} = 6$, s will become our "v"; w_J is the element corresponding to $(st)^3$.

We can obtain the order between pairs of elements by looking at the definition of a blow up: $stst$ has order 3 since $tst = tvt$ and $s = v$ meaning we take $\frac{M_{t,v}}{2}$ and $M_{t,v} = 6$. sw_J has order 2 since $w_J = x \in \{tvt, w_J\}$ and $s = y \in S \backslash J$. Finally, $tstw_J$ has order 2 since $x = w_J$ and $tst = y \in (\{tvt\} \cup J) \backslash \{t\}$. This gives us the group:

$$W' = \langle s, tst, w_j | s^2, (tst)^2, w_J^2, (stst)^3, (sw_J)^2, (tstw_J)^2 \rangle.$$

This group is isomorphic to W yet it has more Coxeter generators! (Remark we could have also obtained a 3-generator group by taking s as our pseudo transposition.)

Results

Now we have found a way of obtaining 3-generator Coxeter presentations of D_n, the immediate question is "how could we obtain them since *Corollary 1* says we couldn't?", and with closer inspection of the formal definition of a Coxeter group, we may realise that we have an element that is not a reflection which fits the criteria of a generator.

Lemma 2: For a Coxeter system $J = \langle s, t | s^2, t^2, (st)^n \rangle$ which is isomorphic to the dihedral group D_n, the element corresponding to the longest word w_J is a rotation by π.

Proof: From *Lemma 1*, we know that $\alpha_{s,u} = \pm\frac{\pi}{n}$.

Fix some axis a passing through the centre of our n –gon. We may now express s as $Ref(\alpha_{a,s})$, to denote a reflection in the line which forms an angle of $\alpha_{a,s}$ with a and t as $Ref(\alpha_{a,t})$ defined similarly. By definition, $\alpha_{a,t} = \alpha_{a,s} \pm \frac{\pi}{n}$ allowing us to rewrite t as $Ref(\alpha_{a,s} \pm \frac{\pi}{n})$.

We may use the following identity:

$$Ref(\theta)Ref(\phi) = Rot(2\theta - 2\phi)$$

Where θ and ϕ are some angles, and $Rot(\alpha)$ denotes a rotation by α.
Substituting:

$$Ref(\alpha_{a,s})Ref\left(\alpha_{a,s} \pm \frac{\pi}{n}\right) = Rot\left(2\alpha_{a,s} - 2\left(\alpha_{a,s} \pm \frac{\pi}{n}\right)\right),$$
$$= Rot\left(2\alpha_{a,s} - 2\alpha_{a,s} \mp \frac{2\pi}{n}\right),$$
$$= Rot\left(\mp\frac{2\pi}{n}\right).$$

Consider the longest word w_J, We can easily identify it as $(st)^{\frac{n}{2}}$. Using the value of st we derived above:

$$w_J = (st)^{\frac{n}{2}},$$
$$= Rot\left(\mp\frac{2\pi}{n}\right)^{\frac{n}{2}},$$
$$= Rot(\mp\frac{2\pi}{n} * \frac{n}{2}),$$
$$= Rot(\pi).$$

as claimed. ■

The blown-up 3-generator dihedral presentation is:

$$D_n = \langle j,k,l | j^2, k^2, l^2, (jk)^{\frac{n}{2}}, (jl)^2, (kl)^2 \rangle$$

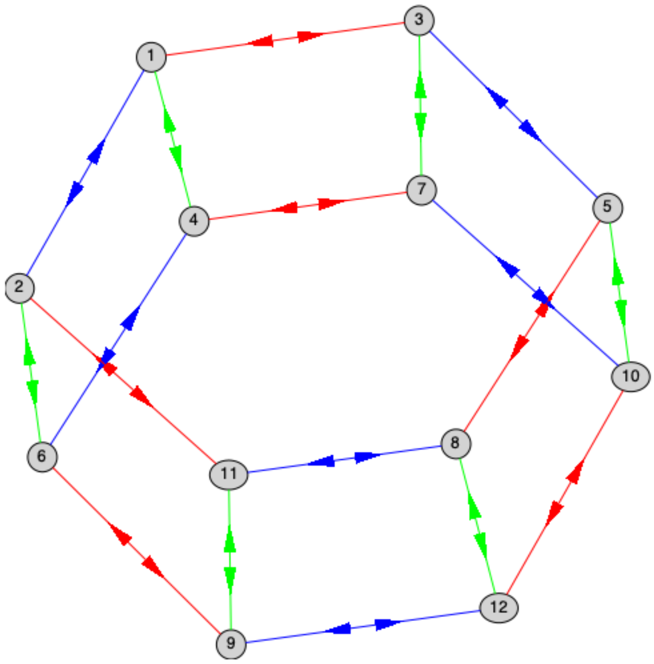
where j, k, l are identified as v, tvt, w_J respectively.

Remark: Due to the condition that the parabolic subgroup must be of type $I_2(2k)$ for $k > 2$ where k is odd, we are only able to blow up D_n if $n = 2k$ subject to the same conditions.

With that in mind, we able to create Cayley graphs of blown-up dihedral groups:

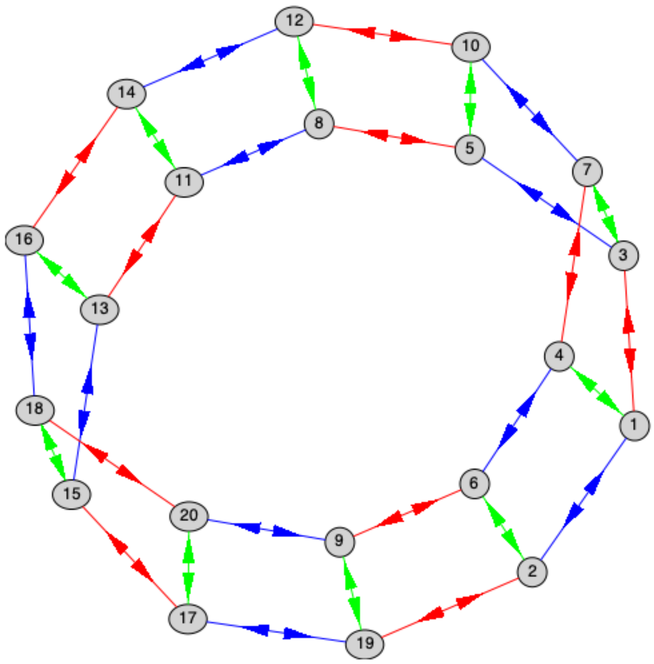
Example 3: To illustrate, take again D_6. We saw above how to blow-up this group; with the generating set $S' = \{j,k,l\}$ and presentation the $W' = \langle j,k,l | j^2, k^2, l^2, (jk)^3, (jl)^2, (kl)^2 \rangle$. We can produce the Cayley graph $\Gamma(W', S')$.

We note that the graph forms a hexagonal prism.

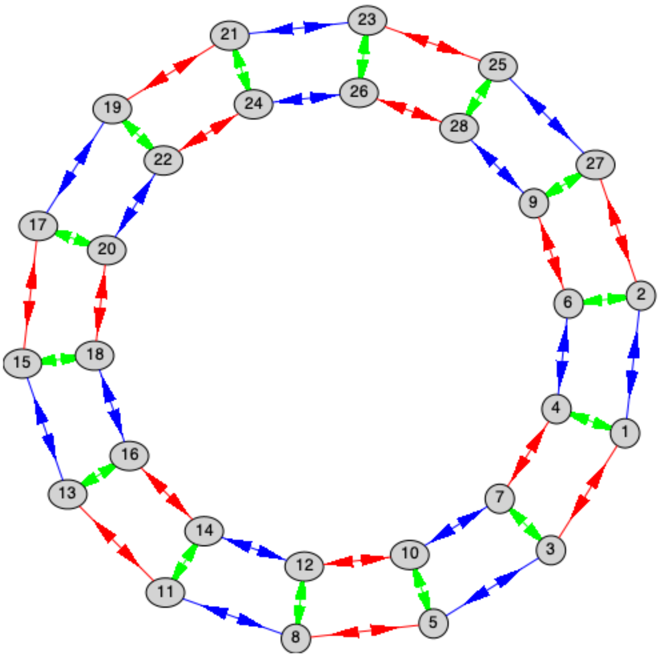


D_n forming a n-gonal prism with the 3-generator Coxeter presentation is a trend that continues, as seen in the following examples:

D_{10} with the generating set $S = \{j, k, l\}$ and the presentation $W' = \langle j, k, l | j^2, k^2, l^2, (jk)^5, (jl)^2, (kl)^2 \rangle$. produces a Cayley graph that forms a decagonal prism.



D_{14} with the generating set $S = \{j, k, l\}$ and the presentation $W' = \langle j, k, l | j^2, k^2, l^2, (jk)^7, (jl)^2, (kl)^2 \rangle$. produces a Cayley graph that forms a dodecagonal prism.



Conclusion

We observe that, with respect to the standard Coxeter presentation of a dihedral group D_n, the Cayley graph for such a presentation is exactly a $2n$ −gon. Turning to the 3-generator Coxeter presentation of a blown-up dihedral group, one can show that the resulting Cayley graph is an n −gonal prism. We have observed over the course of this project that this is a repeating trend among all dihedral groups that can be blown up. Geometrically, it seems including the longest element as a member of the generating set and performing the blow-up produces 'square walls' along the original $2n$-gon. Thus, we conclude that given a Coxeter presentation with two generators which results in Cayley graph forming an $2n$ −sided polygon (where $n \in \mathbb{N}, n > 2$), that it is isomorphic to D_n; if we are given a Coxeter presentation whose Cayley graph forms an n −gonal prism, the presentation is isomorphic to D_n if, and only if, $n = 2k$ where $k \in \mathbb{N}$ is odd and $k > 2$.

We conjecture that this trend continues up to the infinite case of D_∞ - Further work to verify this may achieve this through showing that the blown-up presentation is residually finite and approximating 'square walls' of the prism by taking quotients.

A similar approach for different families of groups may prove fruitful and we highly encourage the reader to give it a try!

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