Comparing Geometries of Reflection Groups

Introduction

Dihedral groups, denoted D_n, are structures that include all possible transformations of a n −sided shape that do not warp its underlying structure; this includes all rotations and reflections. Despite their simplicity, they are significant in many fields including group theory, geometry, and chemistry.

These groups, and many others, can be described with a group presentation. Group presentations take a set of group elements S who's products of powers give every possible group element, and a set of relator, every combination of generators that equal the identity, and provide us with a recipe for creating the group [3]:

$$G = \langle S | R \rangle.$$

Coxeter groups are important examples of groups with a special presentation – every element of the generating set is self-inverse! Due to this, we do not obtain any expressions that are produced by exponentiating one element (barring the generator themselves).

Coxeter Systems

Definition [2]: *Taking M_S as a $n \times n$ symmetric matrix with entries $M_{i,j} \in M_S$*

$$M_{i,j} = \begin{cases} 1, & if\ i = j, \\ n \in \mathbb{N} \cup \{\infty\}, & if\ i \neq j, \end{cases}$$

and $S = \{s_1, s_2, \dots\}$. The group W, defined by the finite presentation

$$W = \left\langle S \middle| \left(s_i s_j\right)^{M_{i,j}}, i, j \in \{1, \dots, n\} \right\rangle,$$

is a Coxeter group attached to a Coxeter system (W, S, M_S).

We can describe D_n in this way with a 2-generator Coxeter system:

$$D_n \cong W = \langle s, t | s^2, t^2, (st)^n \rangle$$

with $S = \{s, t\}$.

Blow-ups

D_n does not immediately admit to a 3-generator Coxeter system – to achieve one, we must identify a pseudo transposition and perform a blow-up:

Definition [1]: *A generator of a Coxeter group t is a pseudotransposition if given a Coxeter system (S, W, M_S), the following conditions are met:*
1. t is contained in a subset $J \subseteq S$ where for any $s \in S \setminus J$ $M_{s,u} = 2$ for all $u \in J$.
2.The parabolic subgroup generated by J,denoted W_J, is of $I_2(2k)$for odd $k > 2$.

Definition/Lemma [1]: *Given a Coxeter system (S, W, M_S) with a pseudotransposition, we define a new generating set:*

$$S' = \{tvt, w_J\} \cup (S \setminus \{t\})$$

where t is our pseudotransposition, $v \in J$ such that $M_{t,v} \neq 2$, and w_J is the longest element in W_J. The new Coxeter matrix $M_{S'}$ has entries

$$M'_{x,y} = \begin{cases} \frac{M_{tv}}{2} & if\ x = tvt, y = v, \\ M_{x,y} & if\ x, y \in S \cap S', \\ M_{v,y} & if\ x = tvt\ and\ y \in J \setminus \{t, v\}, \\ 2 & if\ x = w_J\ and\ y \in (\{tvt\} \cup J) \setminus \{t\}, \\ \infty & if\ x \in \{tvt, w_J\} and\ y \in S \setminus J\ and\ M_{t,y} = \infty, \\ 2 & if\ x \in \{tvt, w_J\} and\ y \in S \setminus J\ and\ M_{t,y} \neq \infty, \end{cases}$$

The Coxeter system $(S', W, M_{S'})$ is called a blow-up of (S, W, M_S).

D_n can be blown up and gives a 3-generator Coxeter system:

$$D_n = \left\langle j, k, l \middle| j^2, k^2, l^2, (jk)^{\frac{n}{2}}, (jl)^2, (kl)^2 \right\rangle$$

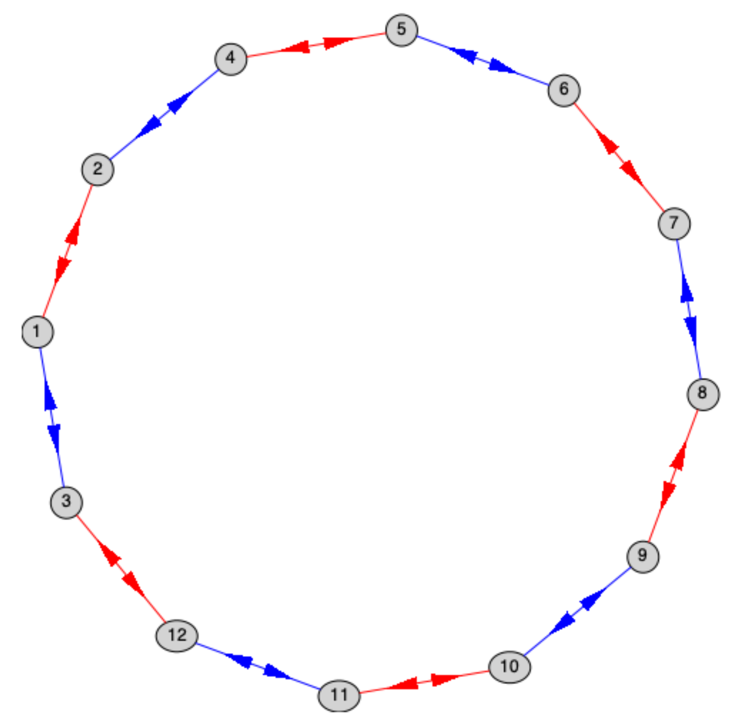
where j, k, l are identified as v, tvt, w_J respectively.

Remark: We can only blow-up D_n if $n = 2k$ where k is odd

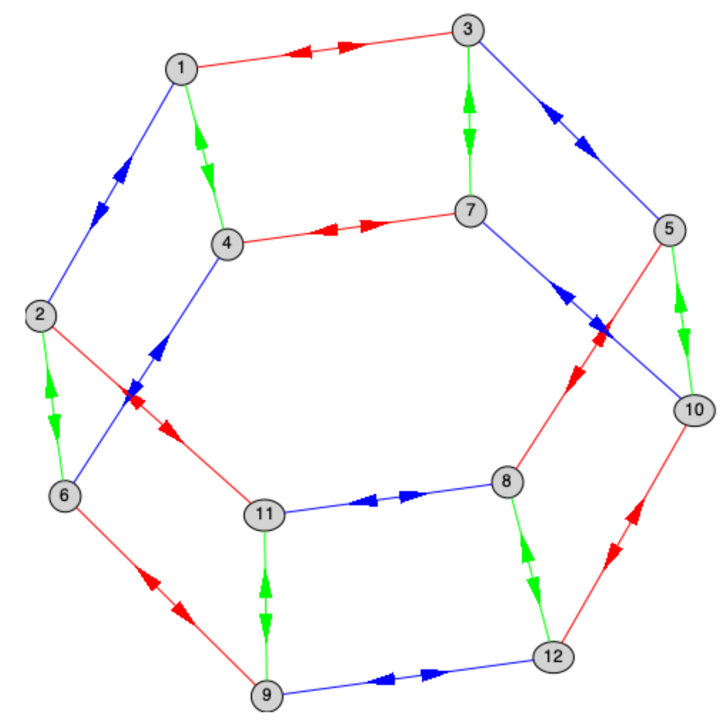
Cayley Graphs

Definition [3]: *For a group G with a symmetric generating set S, the Cayley graph, denoted $\Gamma(G, S)$, is defined as the set of elements of G is the vertex set, and 2 vertices g, h are connected by an edge if there exists some $s \in S$ such that $g = sh$.*

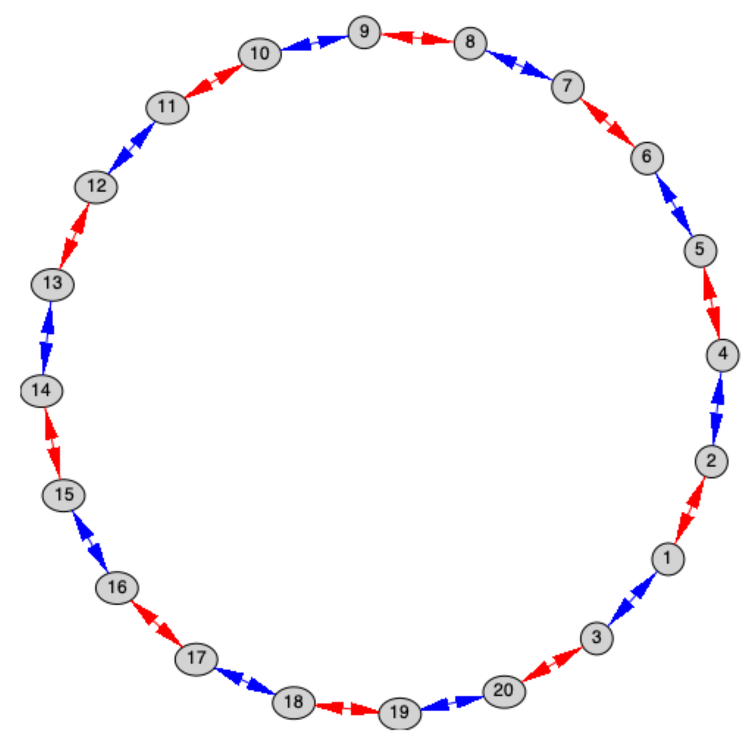
Cayley graphs give us a way of interpreting how group elements are related with respect to the generating set. We can use the Coxeter presentations to produce Cayley graphs of D_n.



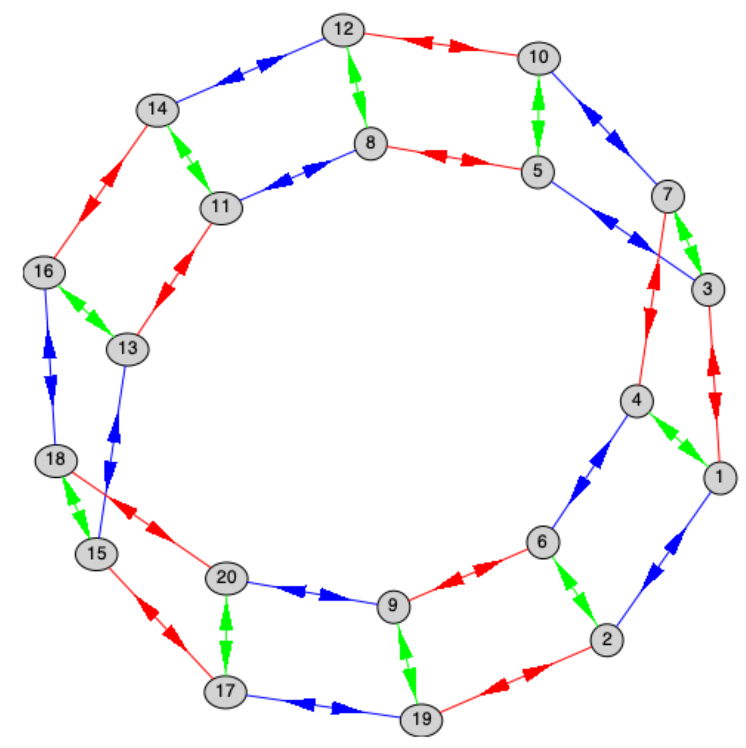
$D_6 = \langle s, t | s^2, t^2, (st)^6 \rangle$



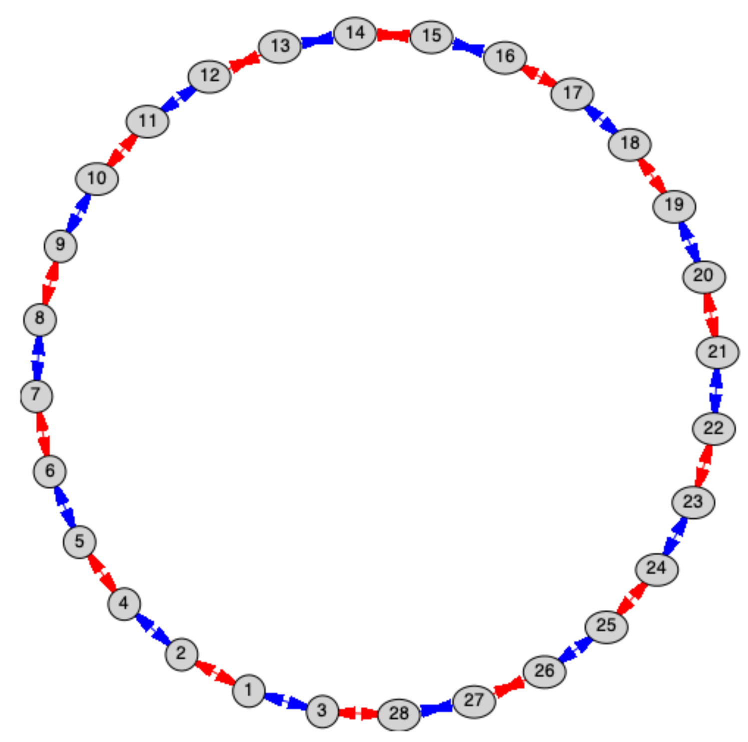
$D_6 = \langle j, k, l | j^2, k^2, l^2, (jk)^3, (jl)^2, (kl)^2 \rangle$



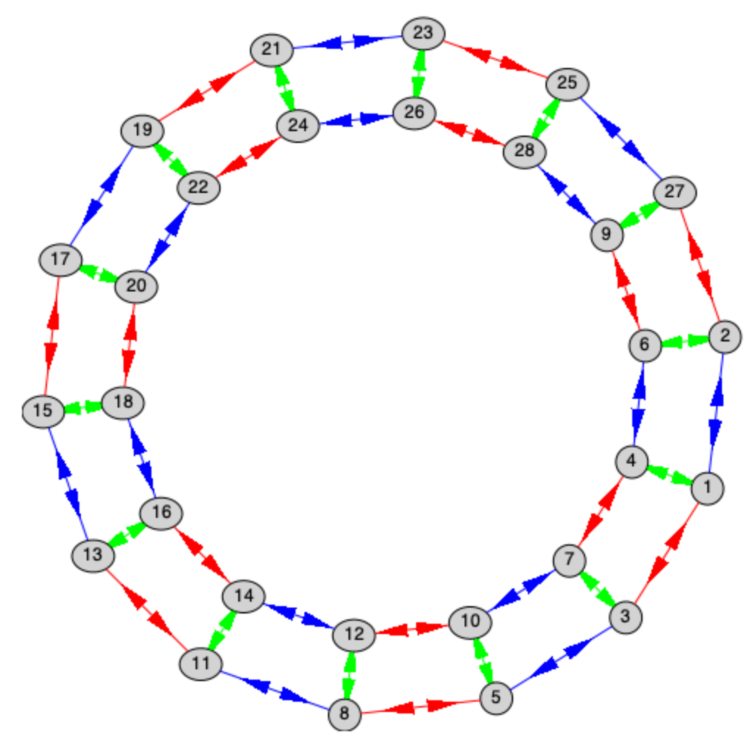
$D_{10} = \langle s, t | s^2, t^2, (st)^{10} \rangle$



$D_{10} = \langle j, k, l | j^2, k^2, l^2, (jk)^5, (jl)^2, (kl)^2 \rangle$



$D_{14} = \langle s, t | s^2, t^2, (st)^{14} \rangle$



$D_{14} = \langle j, k, l | j^2, k^2, l^2, (jk)^7, (jl)^2, (kl)^2 \rangle$

Conclusion

We observe that with a 2-generator Coxeter system isomorphic to D_n, the Cayley graph forms an $2n$-gon. In the instances we can perform a blow up of D_n, we find the 3-generator system produces a Cayley graph that forms an n-gonal prism.

It appears that introducing the element w_J as a generator during the blow-up causes "small squares" to form resulting in a prism.

References

[1] Y. Santos Rego, P. Schwer, *The galaxy of Coxeter groups.* Journal of Algebra, Volume 656, 2024
[2] K. S. Brown, *Buildings*. Springer Science & Business Media, 2013.
[3]C. Löh, *Geometric Group Theory: an introduction*. Springer, 2017.

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