ENGEL CONDITIONS IN BRANCH GROUPS

INTRODUCTION

The study of groups dates back to the ancient Greeks, who studied the symmetries acting on polygons and polyhedra. However, the theory of groups wasn't properly formalised until the mid 19th century by Galois, who studied the permutations of roots of an equation. He found that the structure of the group of permutations of a root determined the solvability of the equation. At the time, Galois studied these groups as permutation groups rather than abstract groups, and it wasn't until the late 19th century when Cayley gave the first definition of an abstract group, which spoke of a set of symbols with an operation. Unfortunately, Cayley's definition lacked a number of important key characteristics of a group such as inverse and identity elements. By the end of the 19th century, Weber developed the modern definition of a group, which is a set of symbols with a binary operation that combines any two elements to make a third which also lies in the set of symbols (closure), whilst also being associative and having identity and inverse elements. For example, a symmetry group of a shape consists of the rotations and reflections of such a shape; if we combine any of these symmetries, we obtain a third symmetry. The symmetries also have inverses which are the exact reverse of the chosen symmetry.

The applications of group theory are very diverse, ranging from cryptography to molecular chemistry. However, the advancements in group theory proceed their applications by about 100 years due to its abstractness. However this highlights how important the study of pure mathematics is in order to develop the future world.

This report focuses on the study of branch groups, in particular the Grigorchuk group, discovered by Grigorchuk in the late 20th century, and the Twisted Twin of the Grigorchuk group, discovered by Bartholdi and Siegenthaler in 2008.

RESULTS

The class of branch groups contains many important groups with interesting properties, however, they were not properly defined until 1999. Informally, a branch group is a subgroup of the automorphism group of a spherically homogeneous rooted tree. They have `tree-like' subgroup structure and act transitively on every level of the tree. The first branch group was constructed by Grigorchuk in the late 20th century, and was the first example of a Burnside group, i.e. an infinite finitely generated torsion group. To this day, many exotic properties of the Grigorchuk group are still being uncovered, which demonstrates the vast applications of this important group.

A group is called Engel if all of its elements are both left and right Engel (see figure 2 for definition). The main goal of Engel theory is to uncover which groups are/are not Engel, and for which groups is the set of right/left Engel elements a subgroup.

In 2015, Bartholdi showed that the Grigorchuk group is not Engel, demonstrating that the set of left Engel elements are exactly the involutions; and therefore is not a subgroup. We wished to uncover whether the Twisted Twin of the Grigorchuk group, discovered by Bartholdi and Siegenthaler, exhibits any similar Engel conditions.

Firstly, we determined the conditions necessary to generalise an example of Bludov to the Twisted Twin of the Grigorchuk group [2]. Bludov's example consisted of showing that the wreath product of the cyclic group K of order 4 with the Grigorchuk group Γ is not Engel. In order to do this, one must choose an quadruple that lies in $Kwr\Gamma$ and continuously take the commutator with $g \in Kwr\Gamma$ until one finds a loop, i.e. we obtain the original quadruple h. This shows that $Kwr\Gamma$ is not Engel, as it demonstrates that $[h, ng] \neq 1$. In an attempt to generalise this example to the Twisted Twin, we did not achieve a full generalisation, however we found certain conditions that would make it easier to do so. For example, we found that we must choose a $h \in KwrG$ (where G denotes the Twisted Twin) such that when we continuously take the commutator with g, we do not lose generators. This is because if our quadruple lies in the subgroup $\langle a, b, c \rangle \leq G$, we can never obtain the generator d.

Next, we looked at a generalisation of Bartholdi's result to the Twisted Twin. In short Bartholdi's example shows that the Grigorchuk group is not Engel. His proof shows that every element not of order 2 in Γ is an Engel element. He showed this by taking an arbitrary element of order 2^e, and we can tale the commutator and it will always be trivial. We could not determine a full generalisation of this, as Bartholdi's proof relied on some GAP calculations, a coding language commonly used in group theory for performing difficult calculations. However, we showed that the appropriate `tools' needed to generalise do indeed work in the context of the Twisted Twin, which gives us a strong reason to believe that we could find an example of a generalisation of Bartholdi's calculation, meaning that the Twisted Twin itself is not Engel. This is an open area for future research.

BRANCH GROUPS AND THE ENGEL PROBLEM

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Let G be a group and let $x, y \in G$. The left Engel commutator of x and y is defined as follows.

$$[x,y] = x^{-1}x^{y}$$
$$[x,2y] = [x,y],y]$$
$$[x,ny] = [x,(n-1)y],y]$$

If $[x,ny] = 1$ for some n, then x is an Engel element.

Figure 2:
Definition of the Engel commutator.

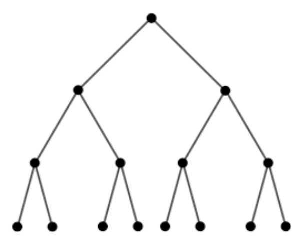

Figure 1:
The binary rooted tree.

REFERENCES AND AKNOWLEDGEMENTS

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