# Algebraic Ping Pong

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***Abstract***

This project studies the ping pong lemma; a theorem that allows one to more easily find free groups. Free groups are the simplest groups as their construction requires no relations. This is opposed to other groups, like the integers modulo n, which have algebraic equations that allow for further simplifications of words. Importantly, free groups have properties associated with them. This then informs us of properties of groups that contain a free group as a subgroup. For example, we can learn about the size of the group, the growth of the group, and the geometry of the group. Using the ping pong lemma, we find a family of free subgroups within the modular group, $SL(2,\mathbb{Z})$ .

***Keywords:*** Ping Pong Lemma, Free Groups, Modular Group, Category Theory

## Introduction

Much of mathematics focuses on the notion of symmetry as a way to study the natures of various patterns. The domain for this study is groups. There are various explanations for this, chief among them being Cayley's Theorem which shows that every group is the same as some set of symmetries (for example, reflection). The power of Group theory can be applied to many fields like chemistry through the symmetries of molecules (Budnikov, Pitirimov, Soldatov and Chuprunov, 2001) and particle physics with the group associated with the standard model, $SU(3) \times SU(2) \times U(1)$(El Naschie, 2007). However, a mathematician's interests tend towards making complicated things more complicated such that they understand a slightly less complicated concept better[1], and thus we look towards how we can study

[1] See "abstract nonsense" (Monastyrsky, 2000)

the nature of groups as mathematical objects. This paper looks at the most generic form of a group given its definition, free groups. We also investigate the process of finding free groups within the modular group, $SL(2,\mathbb{Z})$

## Literature Review

The ping pong lemma is a wonderful tool in group theory, which allows for useful analysis of a given group. Its first great importance comes from Jacques Tits' Free Subgroups in Linear Groups (Tits, 1972), in which Tits proves his alternative: every finitely generated linear group over a characteristic zero field either contain a non-abelian free subgroup or a solvable subgroup of finite index. More recent works show that the lemma still maintains plenty of relevance. Frieden, Gélinas, and Soucy's Cones and Ping-Pong in Three Dimensions applies the lemma to an infinite family of hypergeometric groups to show they are thin (Frieden, Gélinas and Soucy, 2021). While Bashwinger and Zaremsky use the lemma to show non-inner amenability of Higman-Thompson groups (Bashwinger and Zaremsky, 2022). This paper uses many insights contained within Clara Loeh's Geometric Group Theory (Löh, 2018).

## Project Background - Defining $H_n$ Free Groups

We start with two complementary definitions of free groups. The first has less immediate intuition but is ultimately more useful. Whereas the second is the reverse so that it is insightful, but less powerful. This section is based on Clara Löh's book Geometric Group Theory (Löh, 2018).

The aim of this project is to collect knowledge on free groups and present it for a wider audience. Free groups allow mathematicians to understand the "simplest" way a group can be constructed (that is one that has no other restrictions on it). Furthermore, these free groups have a specific structure which can be reflected in other groups that contain free groups. The results of this project should hopefully elucidate what a free group is and the most well-known method for finding them in other groups - thus allowing mathematicians to know more about other groups also.

### Categorical Definition

Category theory is a powerful and rather abstract branch of mathematics. The abstraction is so we can investigate patterns within mathematics without having to worry about which exact mathematical object, like groups, vector spaces or sets, we are dealing with.

The definition in this section does use category theory, however in a very light manner such that one only needs to understand morphisms or maps to be able to grasp it. Without further ado,

**Definition 1**. A free group is a group F containing a set S which has the following universal property. For every group G and every map $\varphi : S \to G$ , there is a unique group homomorphism $\bar{\varphi} : F \to G$ :

The difficulty with this definition, I believe, is the "For every group" part and the correlating group homomorphism. This is very broad, so we will look at some examples and counter examples to see why this part makes a free group free.

**Example 1** (Integers). The set of integers $\mathbb{Z}$ with addition is a group that is freely generated by $\{1\}$. This can be shown by constructing an isomorphism between $\mathbb{Z}$ and the free group on one generator, $F_1$. Given F($\{1\}$) = $F_1$ is the free group on one generator and $\mathbb{Z}$ is the group of integers with addition, then there exists $\varphi : F_1 \to \mathbb{Z}$ which is an isomorphism between these groups. This can be defined as $\forall a \in F_1, \exists n \in \mathbb{Z}, \varphi(a) = n$ , with $a = 1^n$. One can quickly see that this is both injective and surjective, thus making $\mathbb{Z}$ free.

**Counter Example 2** (Finite Groups). Given a group G with $|G| = m \in \mathbb{Z}$. Proving by contradiction, suppose G is a free group, then, from 1 there must be a unique group homomorphism $\varphi$ from G to any group H. Then let $|H| > m$, we see that a morphism cannot be created - let alone a homomorphism. This is apparent when you try build $\varphi$ by connecting each element in G to an element in H, such that for $g_i \in G, \exists! h_i \in H, \varphi(g_i) = h_i \quad \forall i \in \mathbb{Z}_m$, we then have run out of elements in G to connect to H, which contradicts the definition of a morphism. Therefore, the original statement must be false, thus showing finite groups cannot be free.

**Algebraic Definition**

**Definition 2** (Generating Set). A generating set is a set S which contains generators $s \in S$. These generators, along with their inverses $s^{-1} \in \hat{S}$, act like letters to produce words that make up a group $G_S$ through the operation of conjugation.

One example of a word is $a \cdot a \cdot a \cdot a = a^4$ and an important word is the empty word $\varepsilon = w \cdot w^{-1} = w^{-1} \cdot w, \forall w \in G_S$.Another example of a word would be $a \cdot b \cdot a^{-1}$ for some group $G_{\{a,b\}}$ with a generating set $\{a, b\}$. Though in another group, say $H_{\{a,b\}}$, the element $a \cdot b \cdot \ a^{-1} = a \cdot a^{-1} \cdot b = e \cdot b = b$. The difference between H and G is that H has a relation that equates two words.

**Definition 3** (Relation). A relation is an equation consisting of generators. Typically it is a string of generators and their inverses equal to the identity, like $a \cdot b^{-1} = e$ for some group generated by $\{a, b\}$. However sometimes the equation is rearranged, like the relation for abelian generators - $a \cdot b = b \cdot a$ rather than $a \cdot b \cdot a^{-1} \cdot b^{-1} = e$.

This is not yet enough, since we have endless lists of words that are equivalent: $\{a, a \cdot a^{-1} \cdot a, a \cdot a^{-1} \cdot a \cdot a^{-1} \cdot a, a \cdots\}$.

**Definition 4** (Reduced Word). A reduced word is a word which cannot be simplified further. That is, no generators are next to their inverses.

**Definition 5** (Algebraic definition). The set of reduced words from a generating set S, along with conjugation, form a group $F_{red}(S)$, which is freely generated by S. Principally, this group has no relations.

## Methodology - Understanding Free Groups

Within this section, concepts within group theory and properties of groups will be introduced. The purpose of this is to provide a common language when talking about free groups and to show properties that free groups have which are also reflected in groups that contain them.

**Lemma 6** (Uniqueness and rank). *Let S be a set. Then, up to isomorphism, there is at most one group freely generated by S. Therefore, the cardinality of the set S provides a parameter, called rank, to differentiate free groups between each other.*

Essentially, the rank provides us the number of generators of a free group.

**Proposition 7** (Almost all free groups are non-abelian). *The idea that free groups are not abelian makes a lot of sense, since for a group to be abelian there must be a relation like* $\langle \{a, b, ...\} \mid abb^{-1}a^{-1} \rangle$*. However, there is one case where the group is free and abelian. This is the free group over 1 generator, since a generator and its inverse commutes with itself.*

**Definition 8** (Growth function). *Let G be a finitely generated group and let* $S \subset G$ *be a finite generating set of G. Then*

$$\beta_{G,S} : \mathbb{N} \to \mathbb{N}$$

$$r \to |B_r^{G,S}(e)|$$

*is the growth function of G with respect to S.*

***Remark*** *The above definition is mentioned as it can be shown that free groups have exponential growth. Furthermore, a group that contains a subgroup that is free will also have exponential growth.*

**Definition 9** (Virtually Free). *A group G is virtually free if it has a subgroup H, where the index of H is finite and H is free.*

**Further Methodology and Results - Finding Free Groups**

**Theorem 10** (The Ping-Pong Lemma). *Let G be a group generated by elements a and b. Suppose there is a G-action on a set X such that there are non-empty subsets* $A, B \subset X$ *with B not contained in A and for all non-zero integers, n, we have*

$$a^n \cdot B \subset A \quad \text{and} \quad b^n \cdot A \subset B.$$

*Then G is a free group of rank 2 which is freely generated by* $\{a, b\}$.

*Proof.* To prove Theorem 10 true, we must show that there is an isomorphism $\varphi : F_{red}(\{\alpha, \beta\}) \longrightarrow G$ which maps $\{\alpha, \beta\} \mapsto \{a, b\}$.

By the universal property, there exists $\varphi : F_{red}(\{\alpha, \beta\}) \longrightarrow G$ while mapping $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Since G is generated by $\{a, b\}$, $\varphi$ is surjective, as the words in $F_{red}(\{\alpha, \beta\})$ are effectively relabelled from $\{\alpha, \beta\}$ to $\{a, b\}$. For $\varphi$ to be an isomorphism, we need it to also be injective.

Assume for contradiction, $\varphi$ is not injective, we then can say that there exists a reduced word $w \in F_{red}(\{\alpha, \beta\})\backslash\{\epsilon\}$ where $\varphi(w) = e$ - with $e$ being the identity element of G. This is true since if we cannot find such $w$, then because $\varphi$ is not injective there exists $x, y \in F_{red}(\{\alpha, \beta\})\backslash\{\epsilon\}, \text{such that } \varphi(x) = \varphi(y)$. Therefore $\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(x)^{-1} = e$ and so we have a reduced word which maps to the identity of G.

Now we have 4 possibilities for $w$ depending on the first and last letters. For $k \in \mathbb{N}$, $n_0, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{Z}\backslash\{0\}$, then:

1. $w = a^{n_0} \cdot b^{m_1} \cdot a^{n_1} \cdot \cdots \cdot b^{m_k} \cdot a^{n_k}$ - starting and ending with a non-trivial power of a.
2. $w = b^{m_1} \cdot a^{n_1} \cdot b^{m_2} \cdot \cdots \cdot a^{n_{k-1}} \cdot b^{m_k}$ - starting and ending with b.
3. $w = a^{n_0} \cdot b^{m_1} \cdot a^{n_1} \cdot \cdots \cdot a^{n_{k-1}}$ - starting with a, ending with b.
4. $w = b^{m_1} \cdot a^{n_1} \cdot b^{m_2} \cdot \cdots \cdot b^{m_k} \cdot a^{n_k}$ - starting with b, ending with a.

In Case 1, we see the set "ping pong" back and forth,

$$B = e \cdot B = \varphi(w) \cdot B$$

$$= a^{n_0} \cdot b^{m_1} \cdot a^{n_1} \cdot \cdots \cdot b^{m_k} \cdot a^{n_k} \cdot B$$

$$\subset a^{n_0} \cdot b^{m_1} \cdot a^{n_1} \cdot \cdots \cdot b^{m_k} \cdot A$$

$$\subset \cdots$$

$$\subset a^{n_0} \cdot B$$

$$\subset A,$$

leading to a contradiction as B is supposed to be not contained in A.

In Case 2, instead we look at $\alpha w \alpha^{-1}$, which starts and ends with non-trivial powers of $\alpha$. So we can put this through case 1, since $\varphi(\alpha w \alpha^{-1}) = \varphi(\alpha) \cdot \varphi(w) \cdot \varphi(\alpha^{-1}) = \varphi(\alpha) \cdot \varphi(w) \cdot \varphi(\alpha)^{-1} = e$, resulting in another contradiction.

In Case 3, we look at $\alpha^r w \alpha^{-r}$ for some $r \in [\backslash \mathbb{Z}?] \backslash \{0, -n_0\}$, which has $(\alpha^r w \alpha^{-r} = \alpha^{r+n_0} b^{m_1} a^{n_1} \cdots a^{n_{k-1}} b^{m_k} \alpha^{-r}$. Therefore, starting and ending with non-trivial powers of a, thus also resulting in a contradiction after being put through case 1.

In Case 4, if we look at the inverse of $w$ then we have a reduced word starting with a and ending with b, thus fitting into case 3 and producing a contradiction.

Therefore, $\varphi$ is injective and surjective. Thus $\varphi$ is an isomorphism and G is a free group.

**Definition 11** (Modular Group, $SL(2, \mathbb{Z})$). The modular group is a matrix group of $2 \times 2$ matrices with unit determinants and integer entries, written as follows,

$$SL(2, [\backslash \mathbb{Z}?]) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1, a, b, c, d \in [\backslash \mathbb{Z}?] \right\}.$$

**Theorem 12** (Family of free subgroups within $SL(2, \mathbb{Z})$). *Let* $a = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}), (b = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$, *for all* $\alpha, \beta \in [\backslash \mathbb{Z}?] \backslash \{-1, 0, 1\}$. *Then the subgroup of* $SL(2, \mathbb{Z})$ *generated by is a free group of rank 2.*

*Proof.* Since $SL(2, \mathbb{Z})$ acts on $\mathbb{R}^2$ by matrix multiplication on vectors, we consider subsets of $\mathbb{R}^2$ for the roles of A and B. We choose the subsets

$$A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| > |y| \right\} \text{and} B = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| < |y| \right\}$$

as these are non-empty and B is not contained in A. We now look at the $n^{th}$ power of $a$ for all $n \in [\mathbb{Z}?] \setminus \{0\}$ and $(x, y) \in B$,

$$a^n \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \alpha \cdot n \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x + \alpha \cdot n \cdot y \\ y \end{pmatrix}$$

with

$$|x + \alpha \cdot n \cdot y| \geq |\alpha \cdot n \cdot y| - |x|$$

$$\geq 2 \cdot |y| - |x|$$

$$> 2 \cdot |y| - |y| = |y|,$$

so $a^n \cdot B \subset A$. Similarly for $b$, for all $n \in [\mathbb{Z}?] \setminus \{0\}$ and $(x, y) \in A$,

$$b^n \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}^n \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \beta \cdot n & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y + \beta \cdot n \cdot y \end{pmatrix}$$

with

$$|y + \beta \cdot n \cdot x| \geq |\beta \cdot n \cdot x| - |y|$$

$$\geq 2 \cdot |x| - |y|$$

$$> 2 \cdot |x| - |x| = |x|,$$

and so $b^n \cdot A \subset B$, thus making the subgroup generated by $\{a, b\}$ a free group of rank 2.

## UROS Experience

The Undergraduate Research Opportunities Scheme has provided me with the best view of how research works, especially within pure maths. Furthermore, this project has helped me rediscover why I find maths fun. Studying my passion through university I think has dulled it somewhat, since much of what I have had to do was assignments chosen by someone else that I often did not find interesting. I believe I learnt that my curiosity was an issue, because it would cause delay for assignments. UROS, on the other hand, really requires curiosity and proactiveness. My curiosity drove progress in the project.

I thoroughly enjoyed making various scribbles in notebooks on Free groups and trying to check if some elements of the modular group would in fact generate a free subgroup. I did not enjoy as much the struggle of writing everything down fully, but it has been an invaluable lesson.

I believe the availability of schemes like UROS are imperative for a students' research career. Not for the chance they succeed, but for the inevitability that something goes wrong. My experience has been filled with mistakes and deviations, but I was able to make these mistakes in a low-pressure environment. The experience of research can be found no other way than by researching, and these schemes provide students support through equipment, supervisors, or even bursaries, thus making experience much more accessible.

## Conclusion

We have learnt that a free group is a group with no relations. This "simple" definition then leads to learning interesting facts about what a free group can (e.g. abelian in one case) and cannot be (e.g. finite). This project also studied the exponential growth of free groups and the classification of free groups through their rank.

Using the ping pong lemma, we found a family of free subgroups of $SL(2,\mathbb{Z})$. From this theorem, we can therefore deduce further interesting properties of $SL(2,\mathbb{Z})$, like its growth and geometry. The same methods should prove extremely useful to finding more free subgroups, however the subsets of $\mathbb{R}^2$ would change and this proved challenging in my own pursuit of more examples. Further research should try to take advantage of other methods of finding free groups like the Tits Alternative (Tits, 1972), as this shows a linear group must be either virtually solvable or virtually free - allowing one to test the group's solvability to show whether it is free.

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