UNIVERSITY OF LINCOLN

PATTERNS IN NATURE: A MATHEMATICAL VIEW

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The first words

Galileo Galilei
1564 – 1642

Philosophy is written in that great book which ever lies before our eyes — I mean the universe — but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.

Il Saggiatore (1623)

A home garden: the place where we are going to start

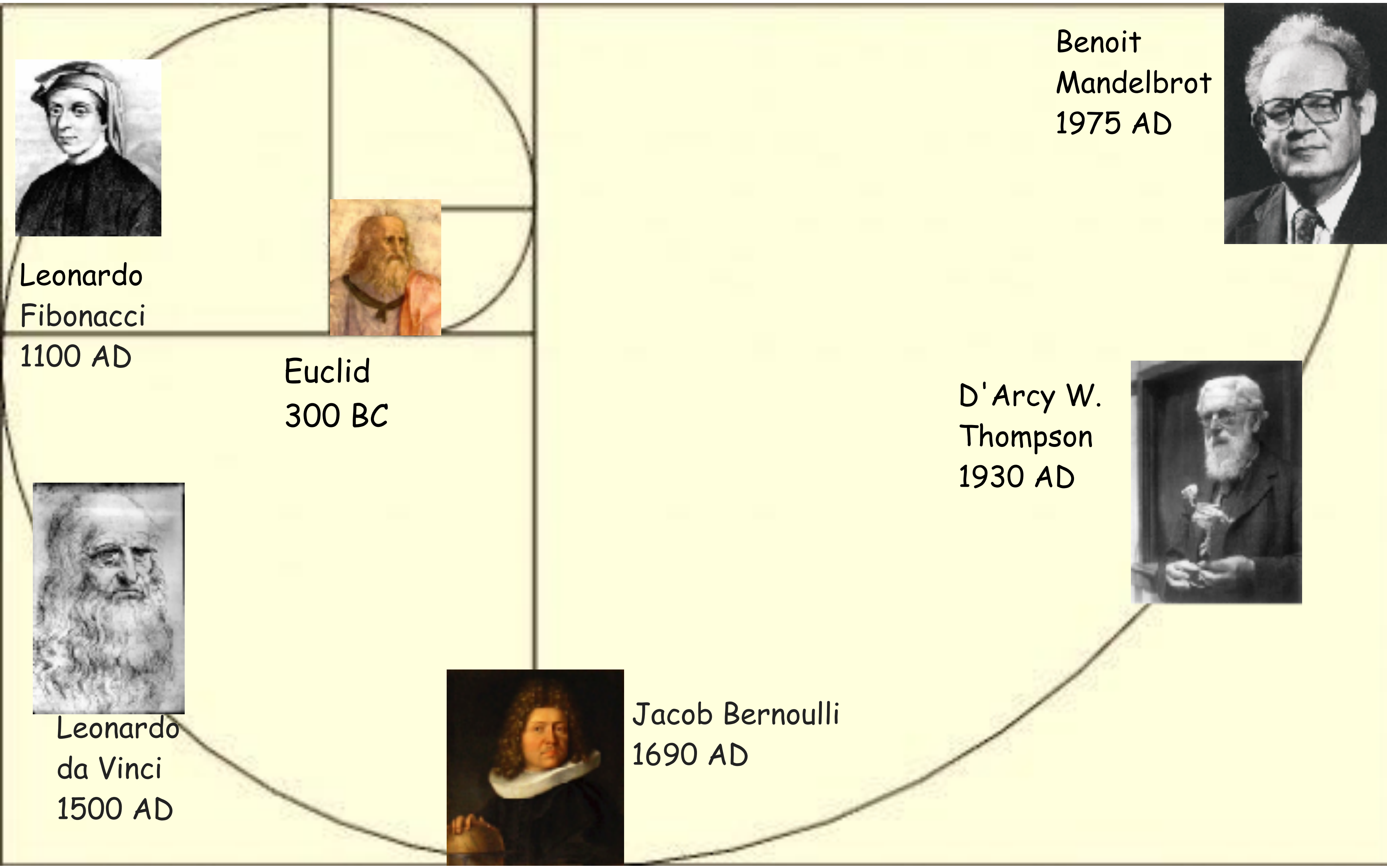
If you look at a garden, many objects before our eyes can be immediately recognised as human artefact and distinguished from natural forms.

Natural shapes are apparently more complex in their structure and surface but indeed they can be mathematically described.

Shape in Nature

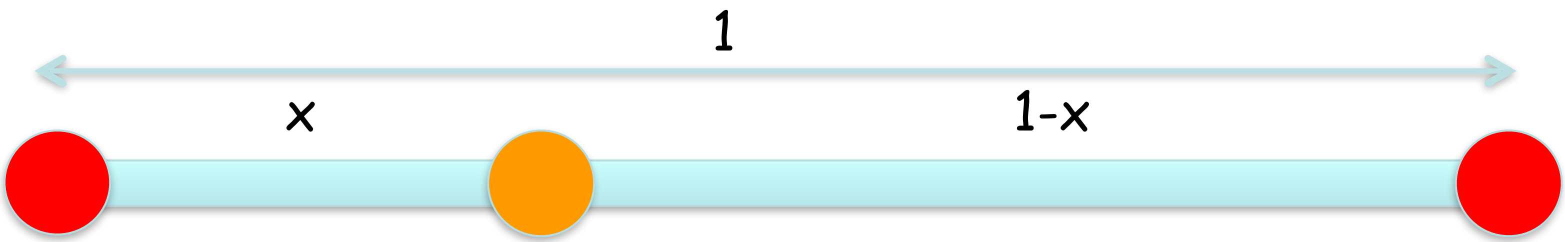
Image source: wikipedia

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html


Leonardo Fibonacci 1100 AD
Euclid 300 BC
Leonardo da Vinci 1500 AD
Jacob Bernoulli 1690 AD
D'Arcy W. Thompson 1930 AD
Benoit Mandelbrot 1975 AD

The golden ratio

Euclid, in *The Elements*, says that the line *AB* is divided *in extreme and mean ratio* by *C* if *AB*:*AC* = *AC*:*CB*.



$$\frac{x}{1} = \frac{1}{(x-1)}$$

$$x^2 - x - 1 = 0$$

The golden ratio

$$x_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$x = \frac{1+\sqrt{5}}{2}$$

= 1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576

28621 35448 62270 52604 62818 90244 97072 07204 18939 11374
84754 08807 53868 91752 12663 38622 23536 93179 31800 60766
72635 44333 89086 59593 95829 05638 32266 13199 28290 26788
06752 08766 89250 17116 96207 03222 10432 16269 54862 62963
13614 43814 97587 01220 34080 58879 54454 74924 61856 95364
86444 92410 44320 77134 49470 49565 84678 85098 74339 44221
25448 77066 47809 15884 60749 98871 24007 65217 05751 79788
34166 25624 94075 89069 70400 02812 10427 62177 11177 78053
15317 14101 17046 66599 14669 79873 17613 56006 70874 80710

The golden ratio

ϕ The symbol used to represent the golden ratio was proposed by the mathematician Mark Barr in honour of Phidias, the Greek sculptor, painter and one of the architects of the Parthenon.

http://www.goddess-athena.org/Museum/Temples/Parthenon/Parthenon_E_from_E1.html

The golden ratio

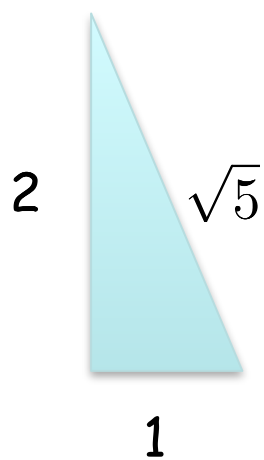
ϕ is an **algebraic number** and also **irrational**

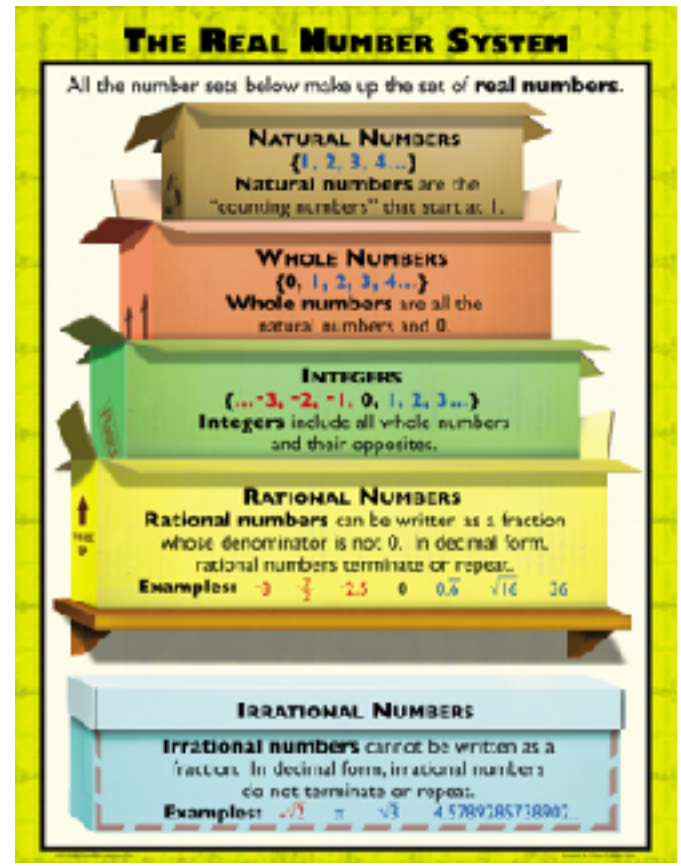
(sequence A001622 in the The On-Line Encyclopedia of Integer Sequences!))

Simple proof

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$2\phi - 1 = \sqrt{5}$$





source: https://www.rainbowresource.com

The golden ratio

It can be also expressed using form starting from:

Substitute the solution: $$(\phi)^2 - (\phi) - 1 = 0$$

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi = 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$\phi = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\phi}}}$$

$$\phi = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\phi}}}}$$

The golden ratio

Continued fraction representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}}}$$

or in more compact way as:

$$\phi=[1;1,1,1,1,1,1,1,1,1,1,1,\ldots]$$

Continued root representation

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}}$$

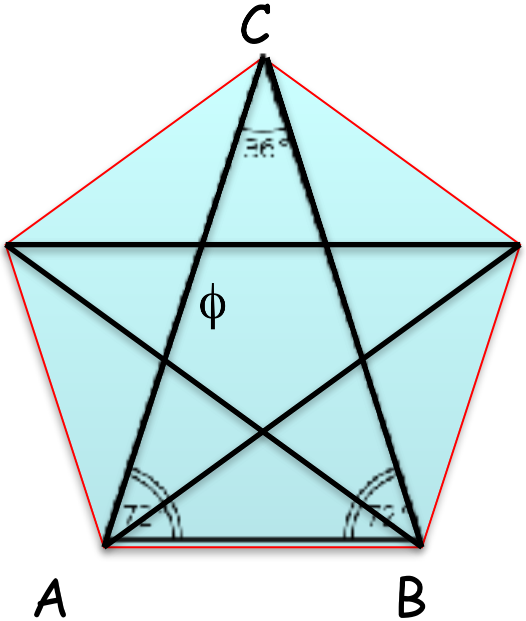
Q1

Geometry and golden ratio

Pentagonal symmetry is common in nature

In the pentagon we find again ϕ

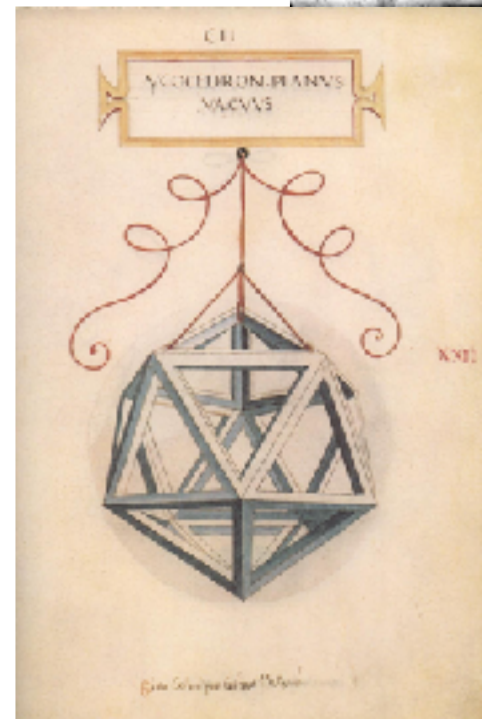
$$\phi = \frac{\overline{BC}}{\overline{AB}}$$



$$\cos 36^0 = 1 - \frac{1}{2\phi^2}$$

Golden ratio in 3d Space
Icosahedron

The golden ratio appear in this polyhedron in numerous way.



Regular faces:	20
Vertices:	12
Edges	30

The 12 vertices have coordinates that are **permutations** of this set:

$(0, \pm 1, \pm\phi)$
$(\pm 1, \pm\phi, 0)$
$(\pm\phi, 0, \pm 1)$

$$V = \frac{5}{6}\phi^2 = \frac{5}{12}(3+\sqrt{5}) \approx 2.18$$

Dodecahedron is the dual polyhedron of icosahedron

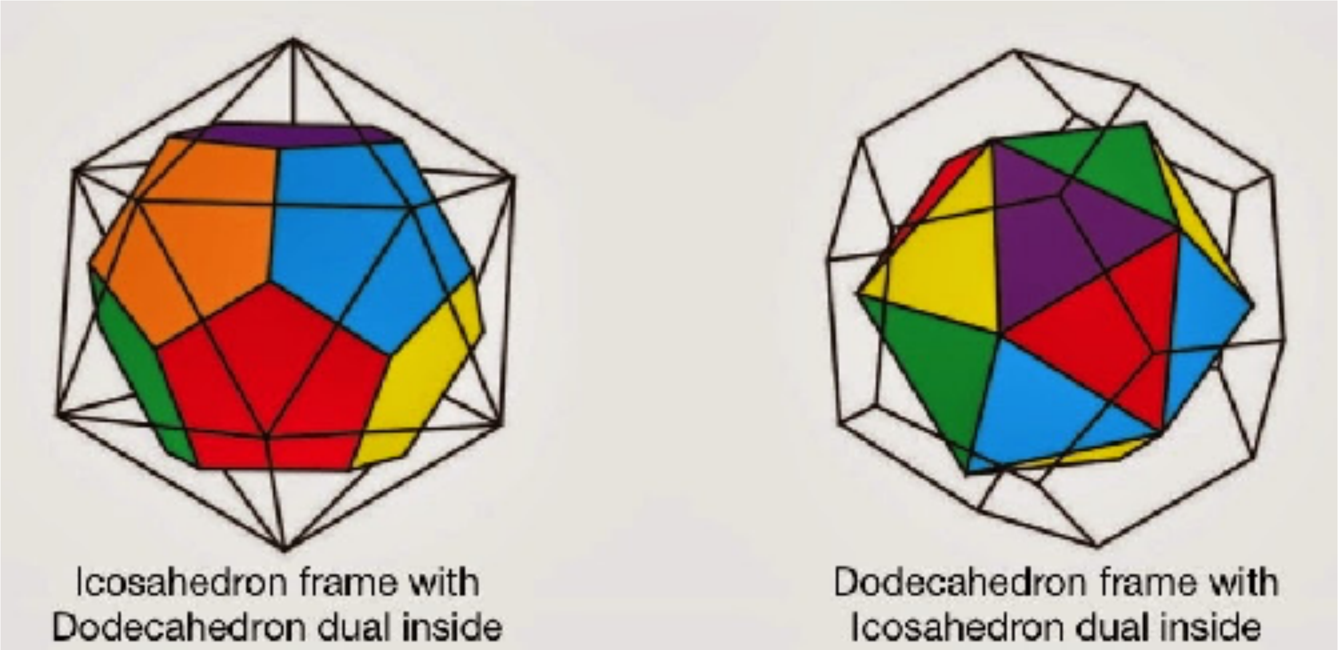
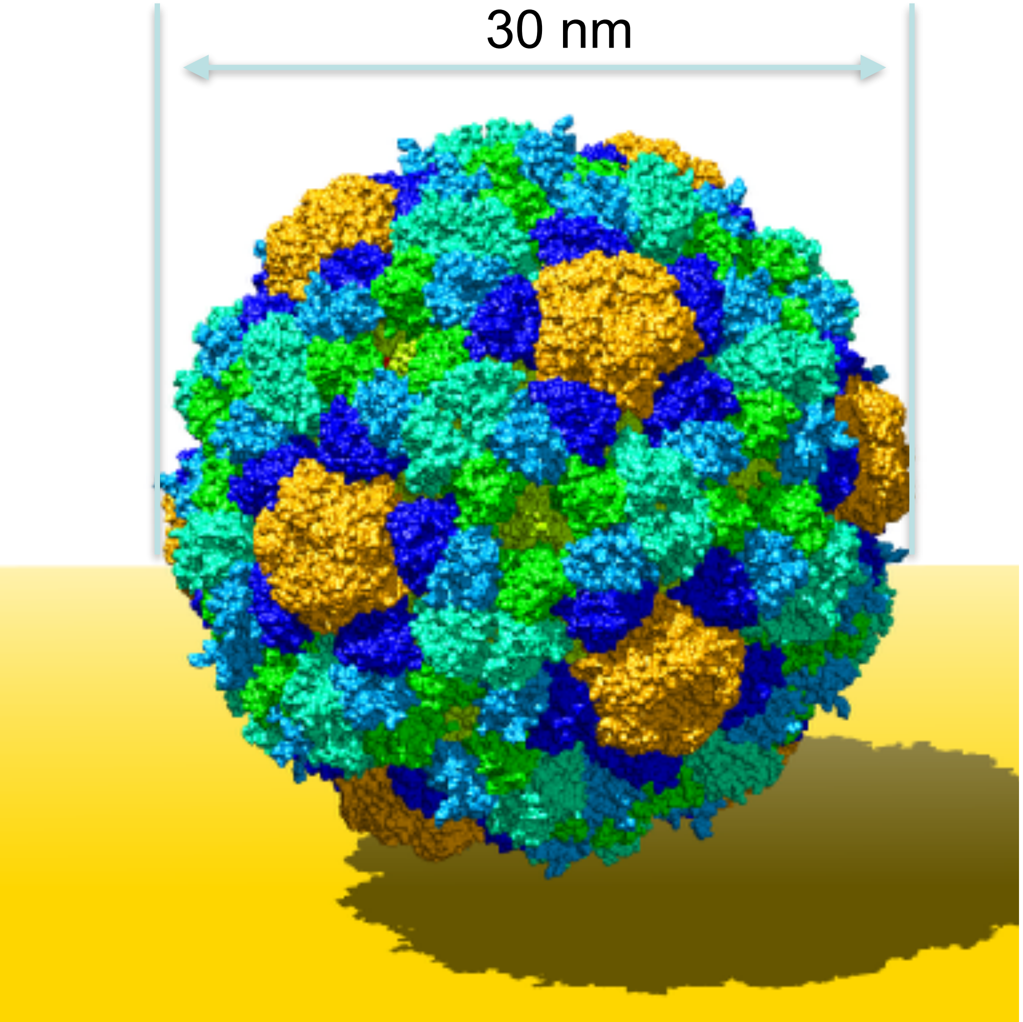


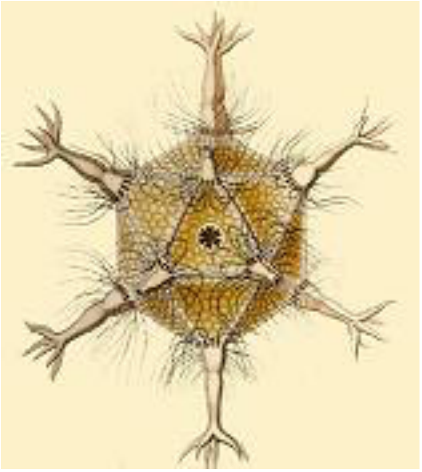
Image source: http://makerhome.blogspot.hr)

and the dodecahedron is even more stuffed of golden ratio relation!

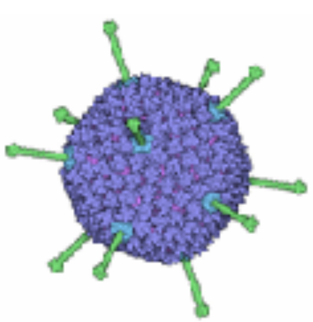
Golden ratio in 3d Space
Icosahedron in Nature

POLIO VIRUS

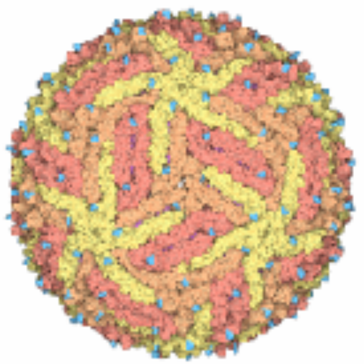




Circogonia icosahedra,
a species of Radiolaria,
shaped like a regular icosahedron

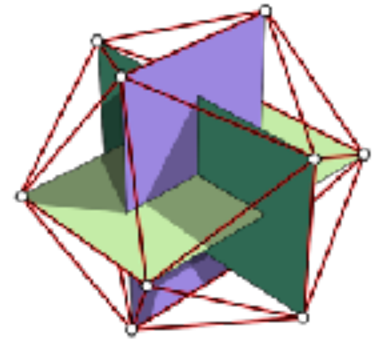


Adenovirus

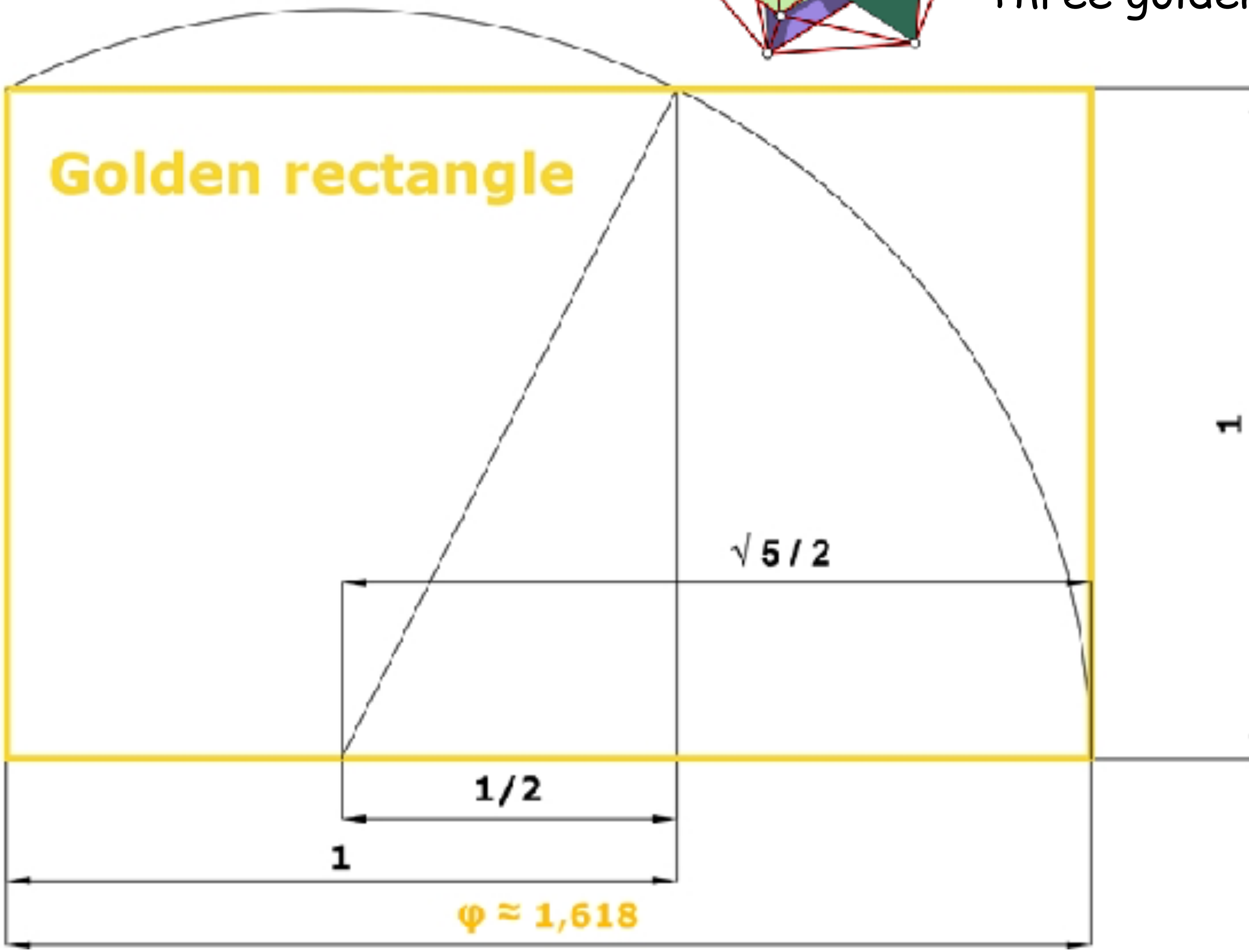


Zika virus

Geometry and golden ratio



Three golden rectangles



The golden spiral is a special logarithm spiral

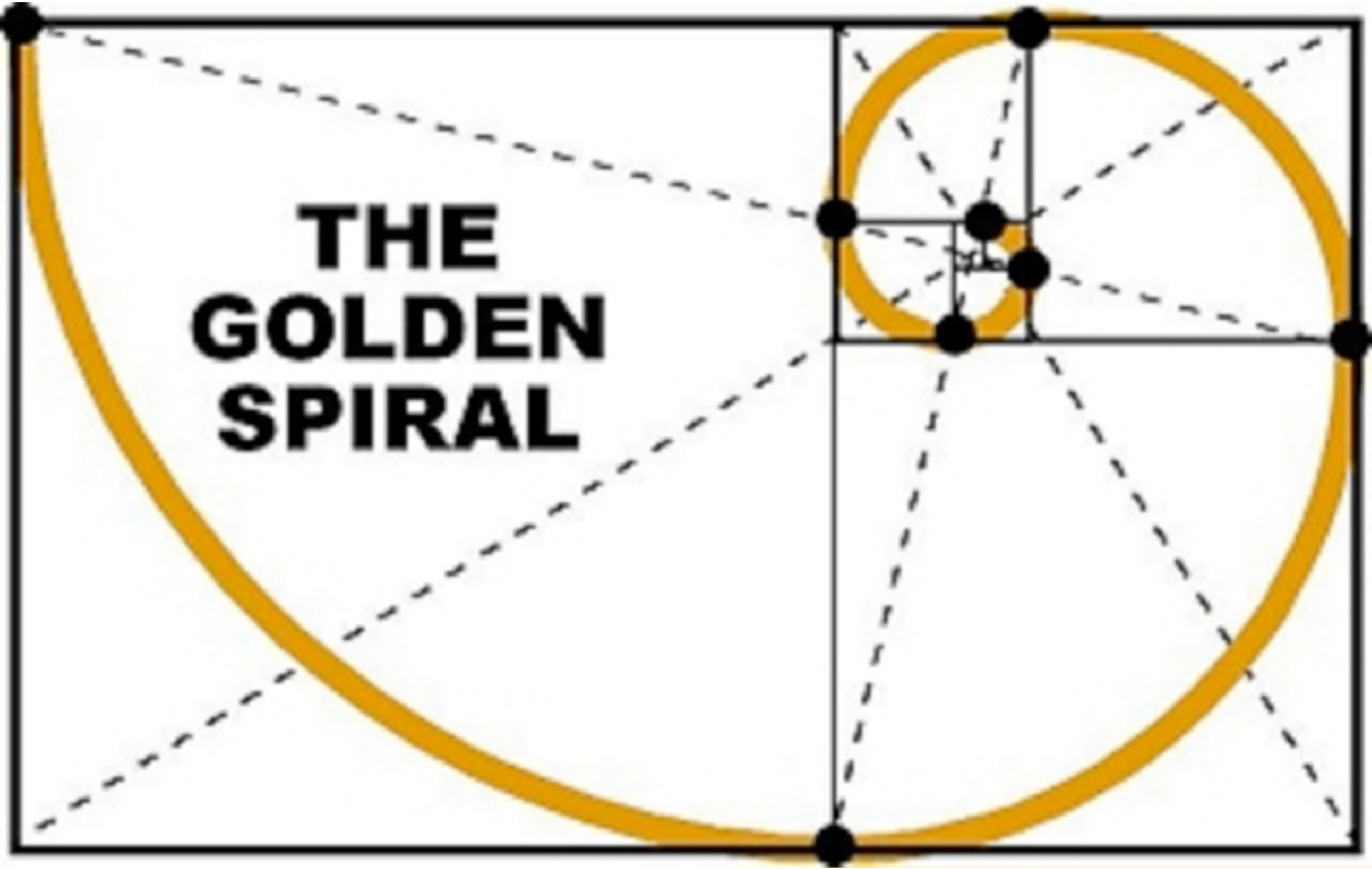


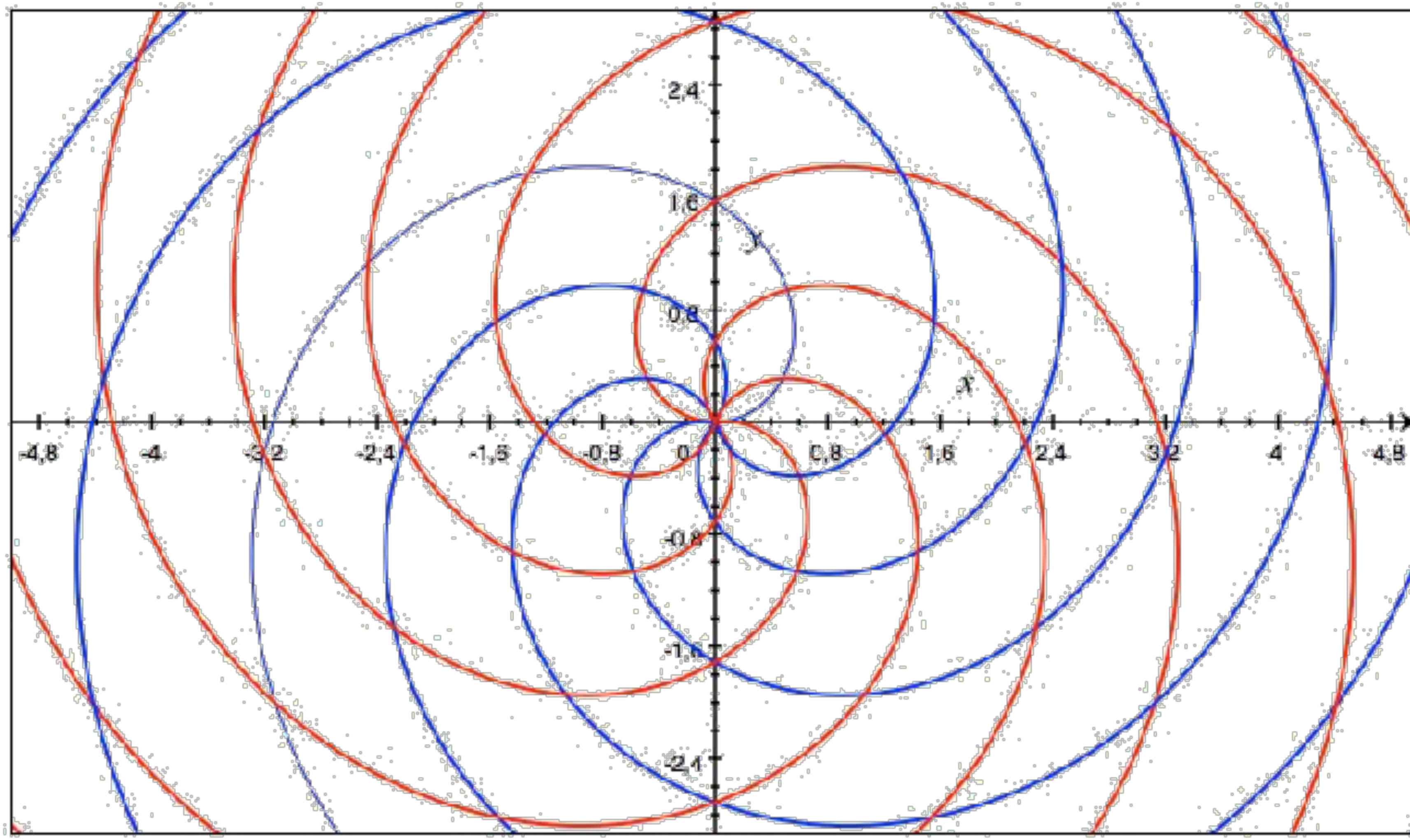
Image source: https://watercolorpainting.com/composition-golden-spiral/

Logarithm Spiral in Nature

Spira mirabilis
of Jacob Bernoulli
(1655-1705)

Image source: wikipedia

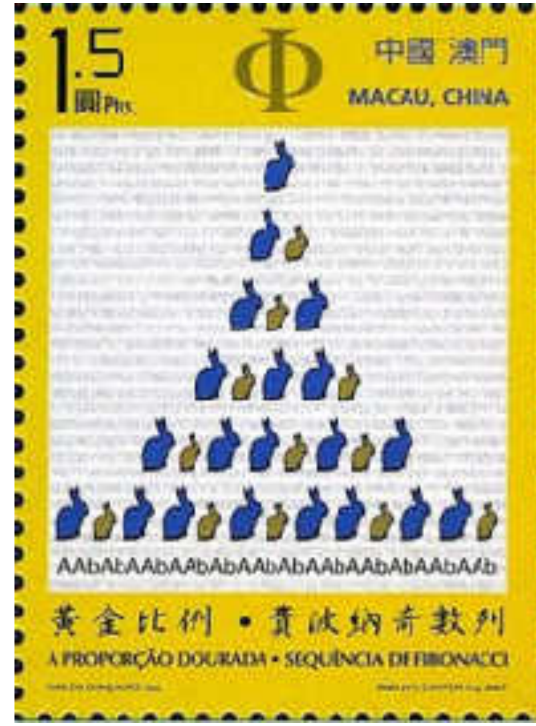
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html



2,4
1,6
0,8
-0,8
-1,6
-2,4
y
x
-4,8
-4
-3,2
-2,4
-1,6
-0,8
0
0,8
1,6
2,4
3,2
4
4,8

Fibonacci Numbers

(sequence A000045 in the The On-Line Encyclopedia of Integer Sequences)



The Fibonacci numbers give the number of pairs of rabbits months after a single pair begins breeding (and newly born bunnies are assumed to begin breeding when they are two months old), as first described by Leonardo of Pisa (also known as Fibonacci) in his book *Liber Abaci.*

$$F_0 = 1$$
$$F_1 = 1$$
$$F_k = F_{k-2} + F_{k-1}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, ...

Leonardo
Fibonacci
1100 AD

Some interesting properties of the Fibonacci Sequence

The sum of the first n number is give by the value of the number (n+1)-1

0, 1, 1, 2, 3, 5,   8,   **13**
n, n+1, n+2
0+1+1+2+3+5=12 —> (n+2)-1=13-1

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, **89**
0+1+1+2+3+5+8+13+21+34=88

Q2

Fibonacci numbers and the golden ratio

1/1 = 1,
2/1 = 2,
3/2 = 1·5,
5/3 = 1·666...,
8/5 = 1·6,
13/8 = 1·625,
21/13 = 1·61538...

Powers of the golden number

$$\phi^2 = \phi + 1$$

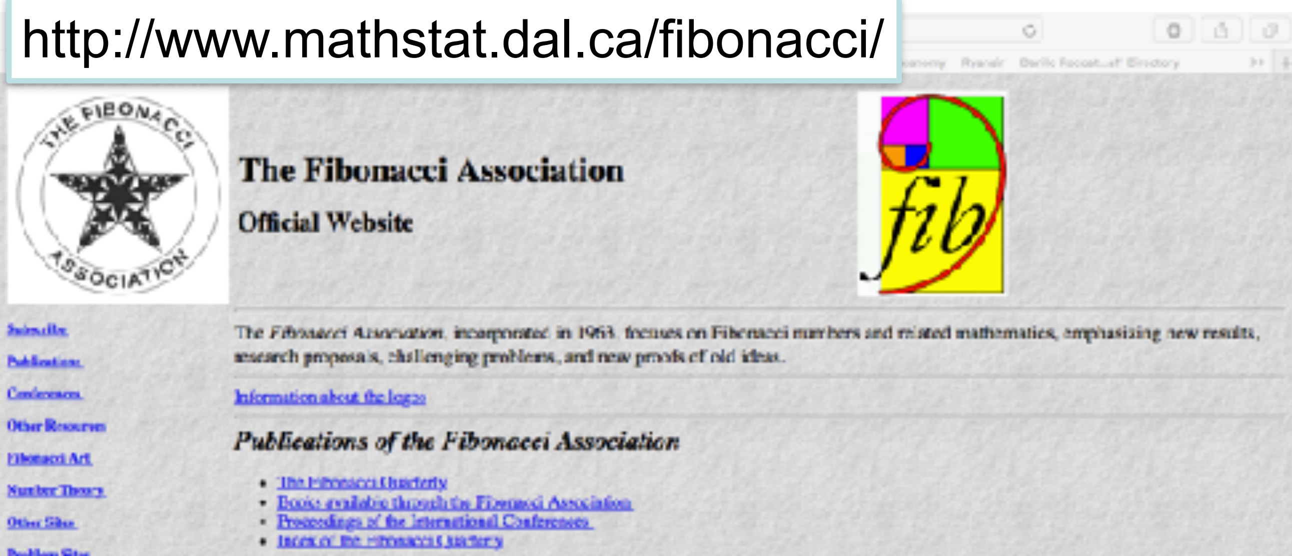
$$\phi^3 = \phi + \phi^2 = 2\phi + 1$$

$$\phi^4 = 2\phi^2 + \phi = 3\phi + 2$$

$$\dots$$

$$\phi^n = F_k\phi + F_{k-1}$$

http://www.mathstat.dal.ca/fibonacci/



Fibonacci in Nature

3

5

8

13



Fibonacci vs Golden spiral

Q3

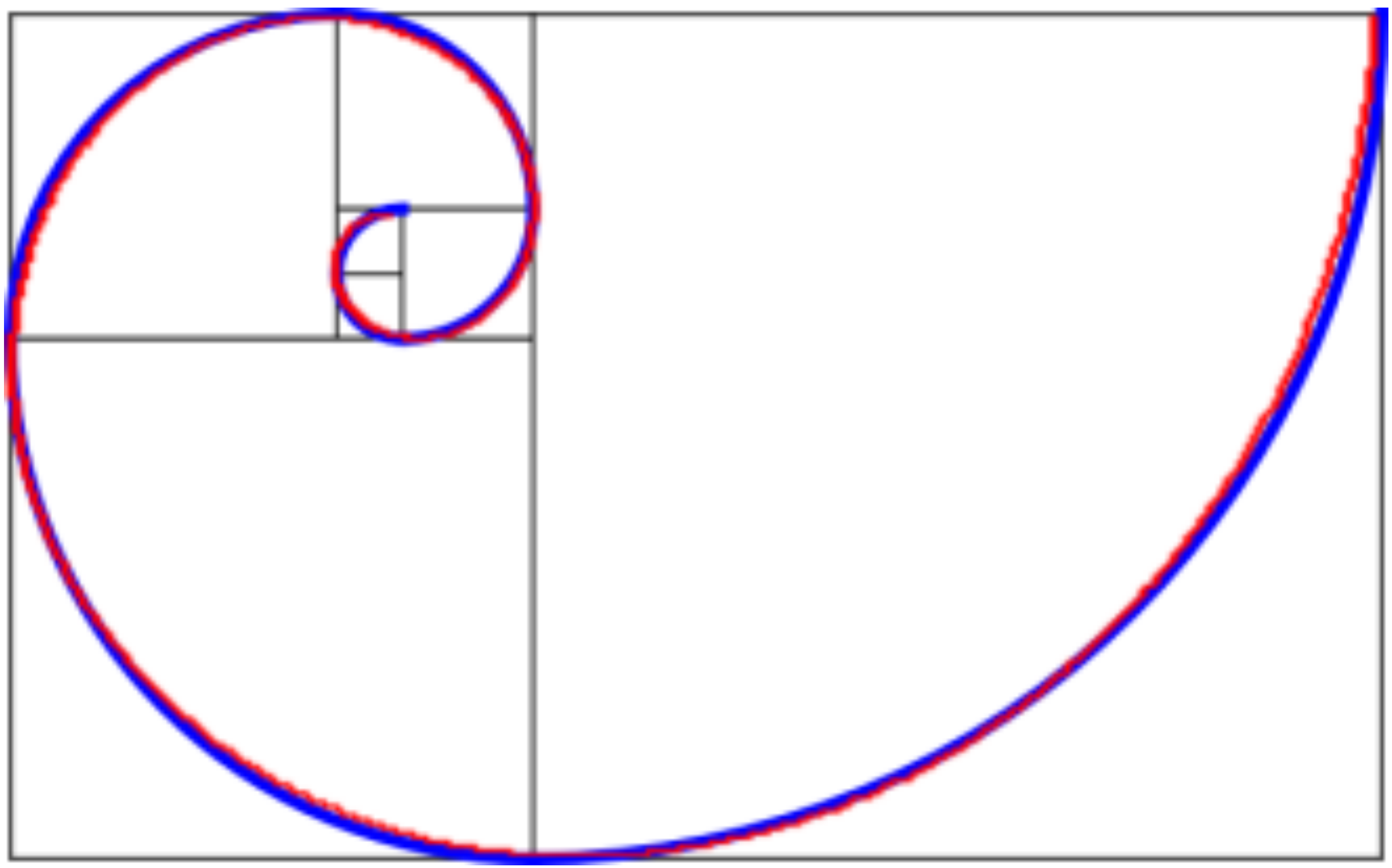


Image source http://yozh.org/2010/11/11/nature-by-numbers/

FRACTALS EVERYWHERE

The Fern plant

The property showed in the previous slide

is called

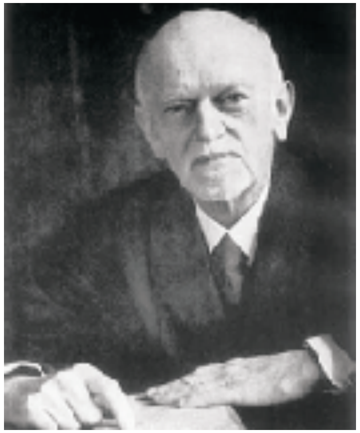
“self- similarity under scaling”

and it is the key to describe

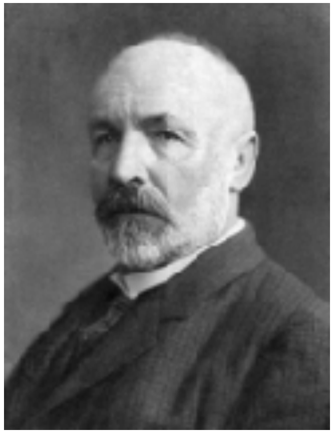
the complexity of Nature…

At the beginning of the last century mathematicians started to explore the self similarity by exploring the properties of peculiar geometric objects

Giuseppe Peano
(1858-1932)



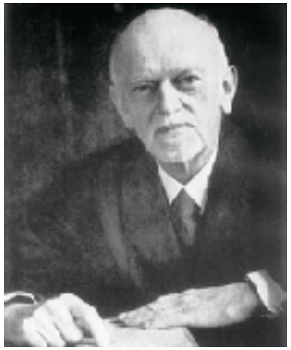
David Hilbert
(1862-1943)



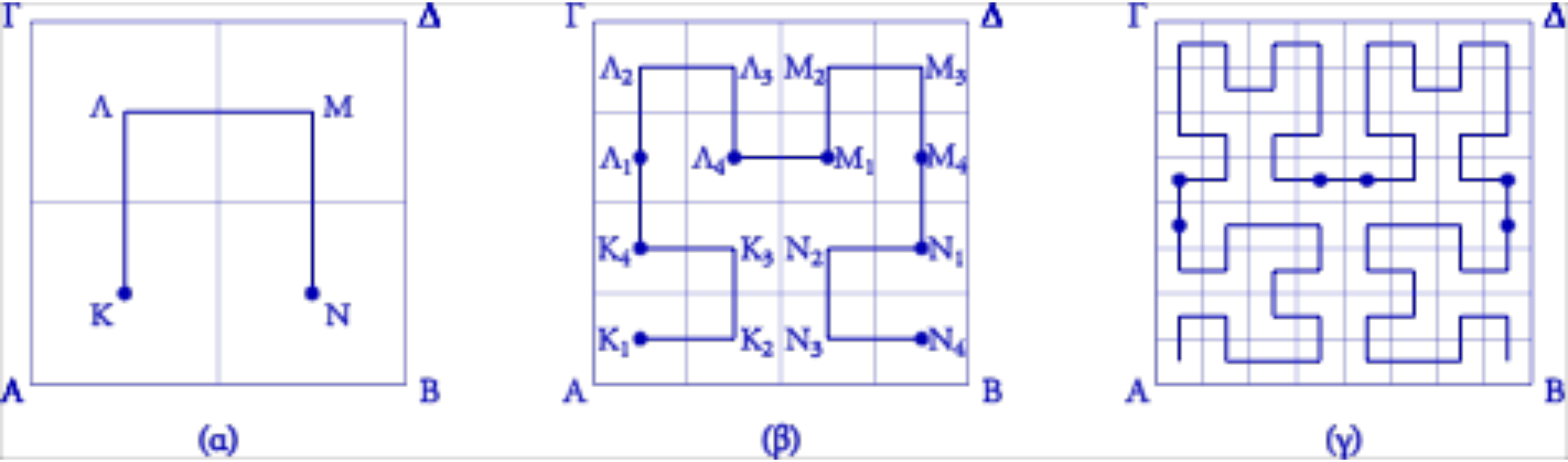
George Cantor
(1845 - 1918)

Niels Fabian Helge von Koch
(1870 - 924)

Hilbert showed how we can construct a curve that fully covers a plane



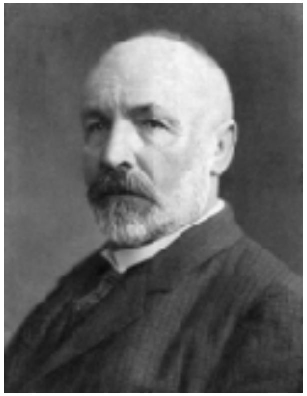
David Hilbert (1862-1943)



If increase the number of square making them smaller and smaller, we notice that the Hilbert's curve seems to cover the plane. However, by its construct it takes for each square only a tiny amount of the possible points.

What is the dimension of the curve? One or two ?

Other simple models of self-similar curves



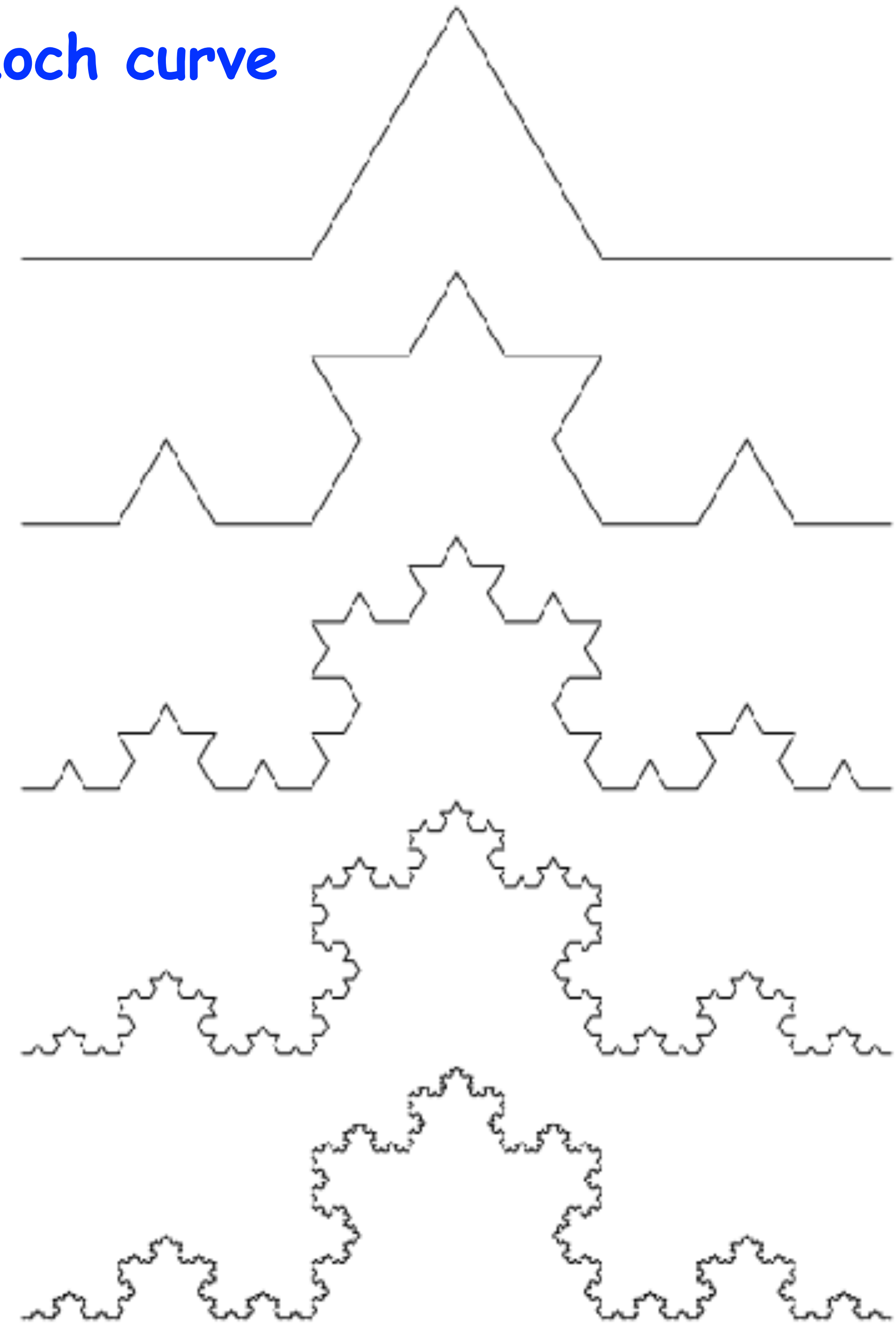
were introduced by

Georg Cantor

and

Niels Fabian Helge von Koch

Koch curve



Starting form the upper curve, we add the same shape to each segment thus creating a mathematical "coastline", of greater and greater complexity

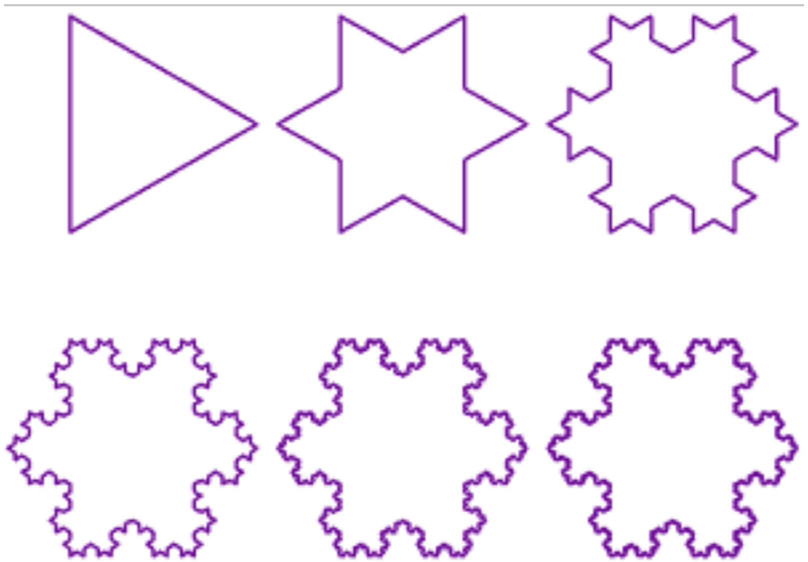
$$(1/3)^{\nu}, \quad \nu = 1, 2, 3, \ldots.$$

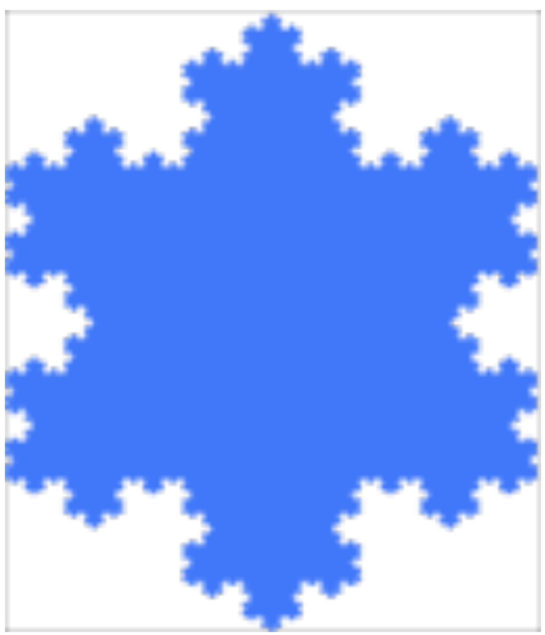
This is the famous Koch curve,

which exist in a mathematical universe of fractional dimension

D = 1.26,

greater than 1 but smaller than 2!





If we use a triangle the island of

Koch (or is it a

snowflake?), which would

have of course finite

area, but…

if you wanted to walk

along its coastline you

would never finish, since

its length is infinite!

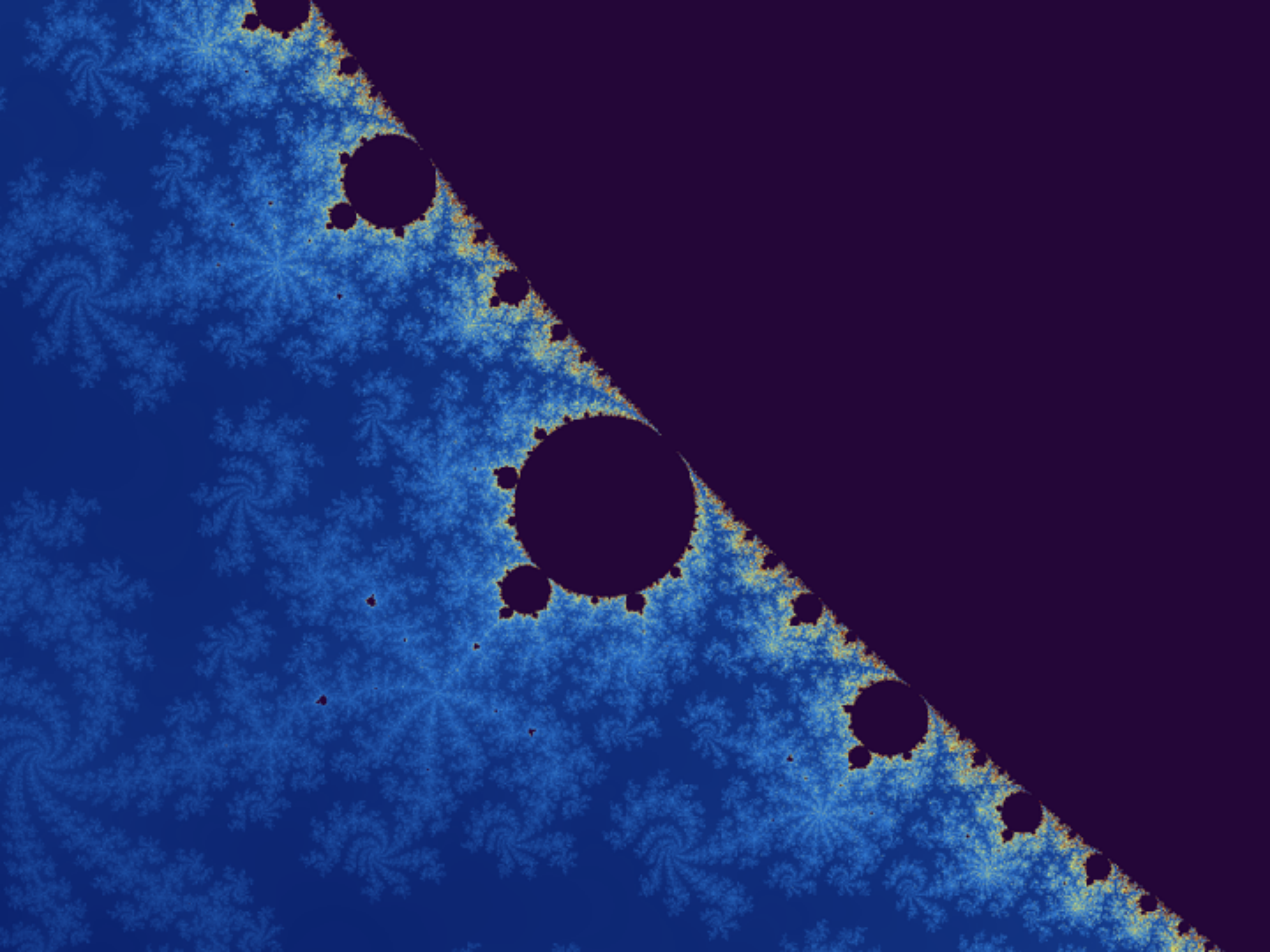
All these objects, which are characterised by «self – similarity under scaling» and generally have non integer dimension

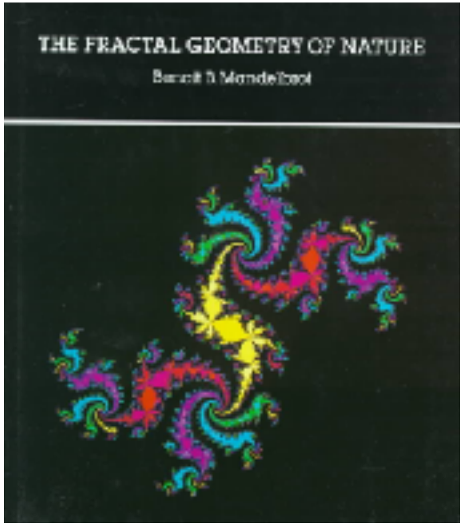
are called FRACTALS.

To understand them we need a new kind of Geometry called Fractal Geometry, that was first introduced by the French mathematician Benoit Mandelbrot in 1970.

He come across to a "mysterious mathematical island" located in the imaginary space now called

the Mandelbrot set where the self-similarity goes beyond the imagination.





Mandelbrot in his famous book

"The Fractal Geometry of Nature", in the late 1970's

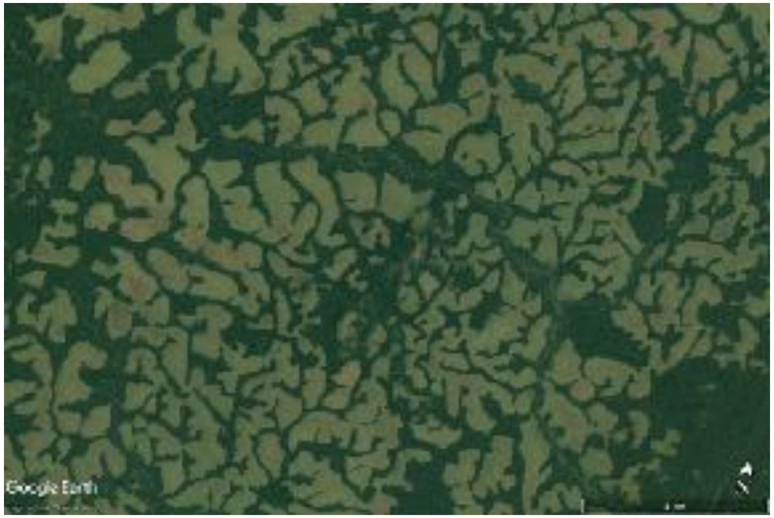
also discuss about the coastline

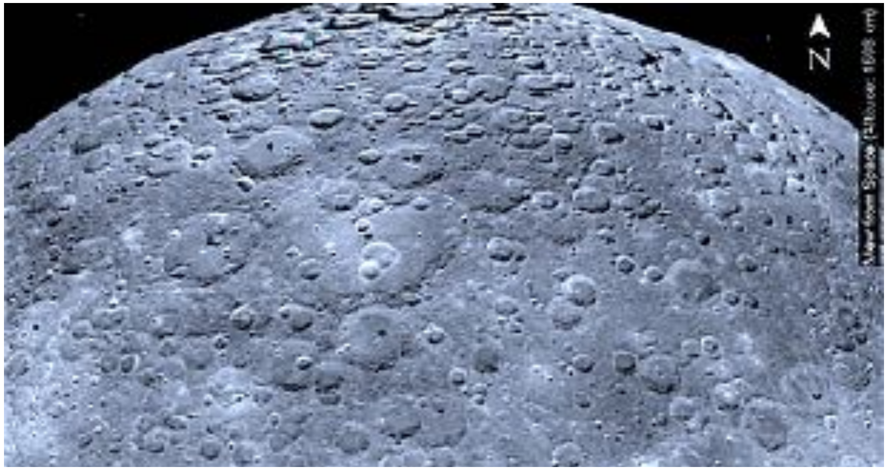
of another famous island ...


Image © 2017 Getmapping plc
Google Earth

The Fractal Geometry of Nature

The Fractal Geometry of Nature



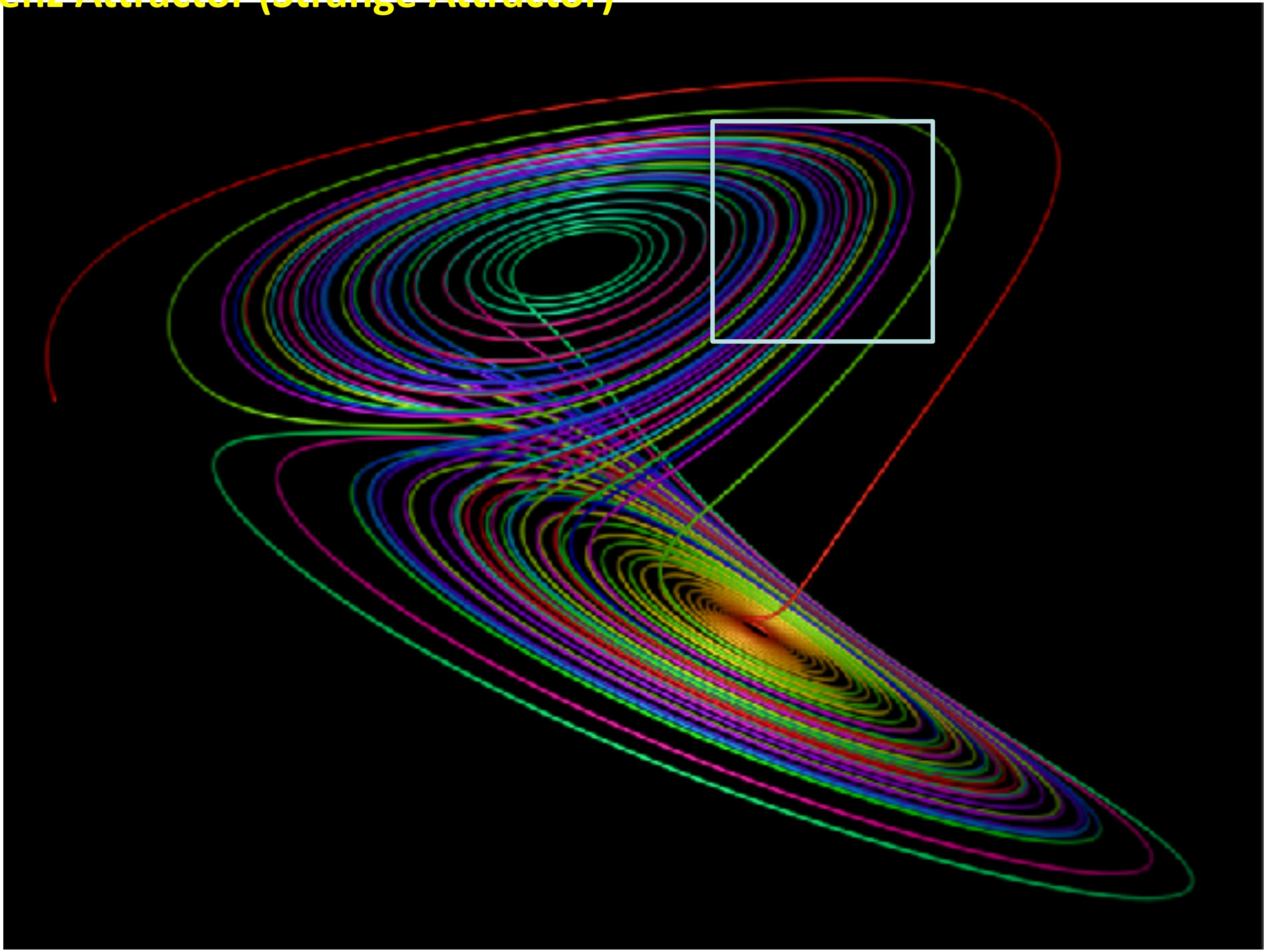


The Fractal Geometry in Art

Autumn Rhythm, 1950, oil on canvas, 266.7 cm by 525.8 cm
Jackson Pollock (1912-1956), American painter, major figure of abstract expressionist movement

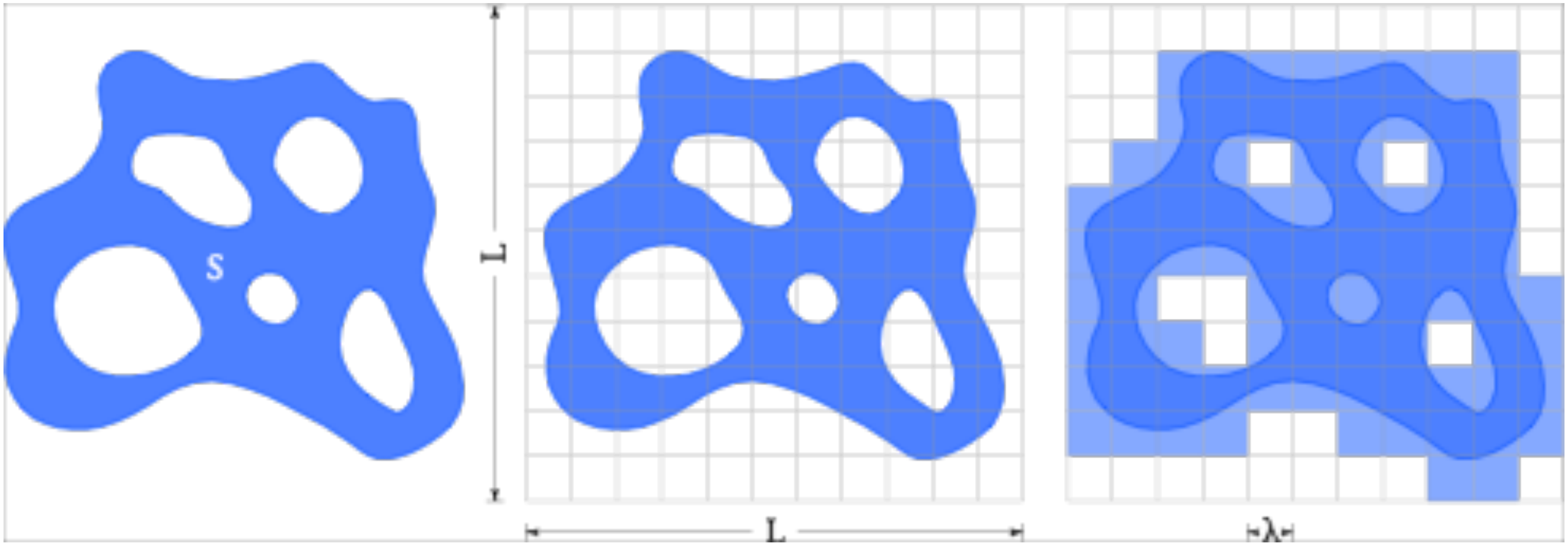
The Fractal Geometry in Art

The Lorenz Attractor (Strange Attractor)



How do we calculate a fractal dimension

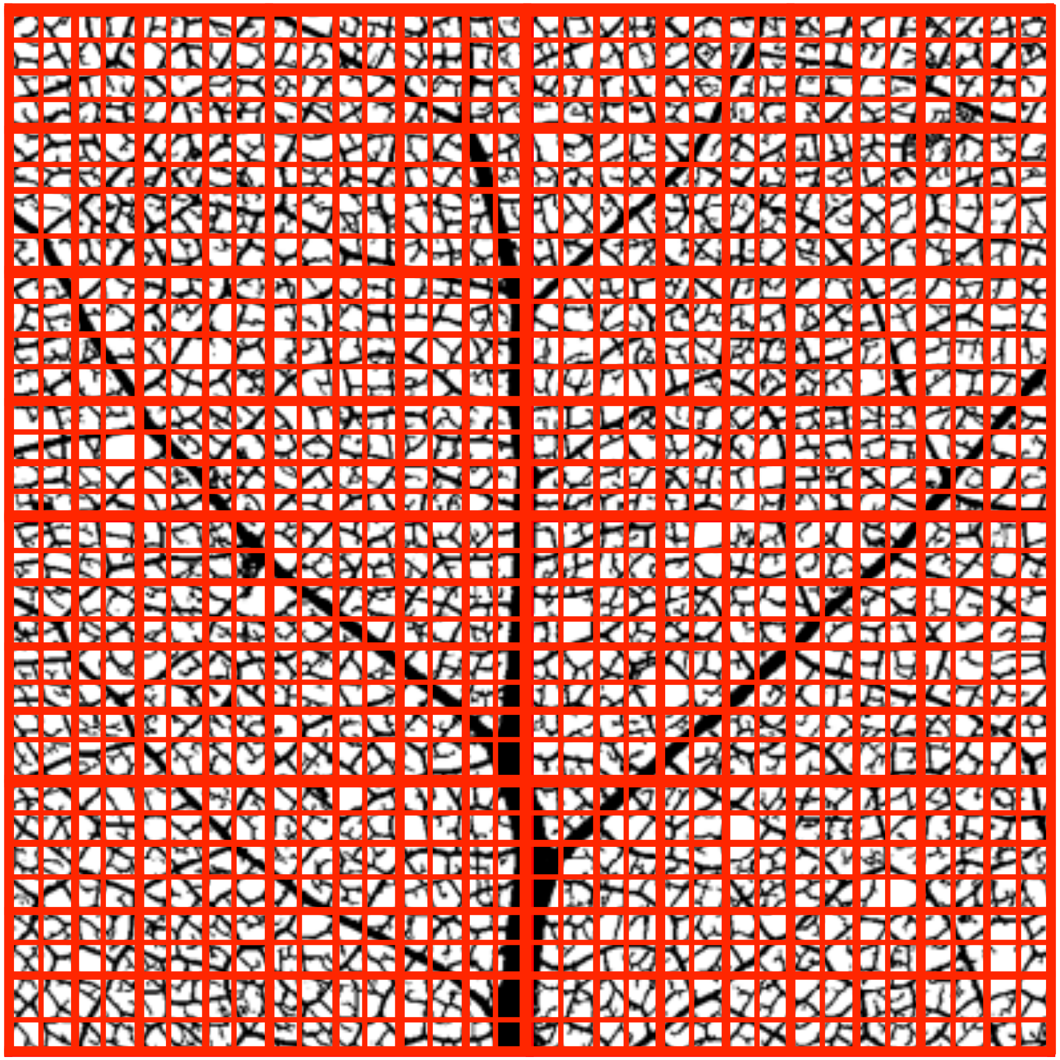
Let us divide the space in which our object is embedded into "boxes" of side ʌ and count the number of "boxes" N(ʌ), which contain at least one point of the object.

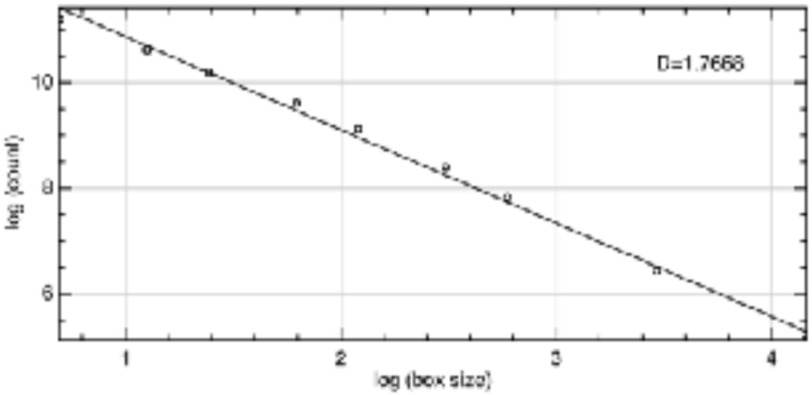


The dimension of our object is the unique exponent D, for which the "measure" of our set $M = N(\lambda)\lambda^D$ is finite, i.e

$$D = \frac{\log N(\lambda)}{\log(1/\lambda)}$$

in the limit ʌ →0 and N(ʌ) → infinity.

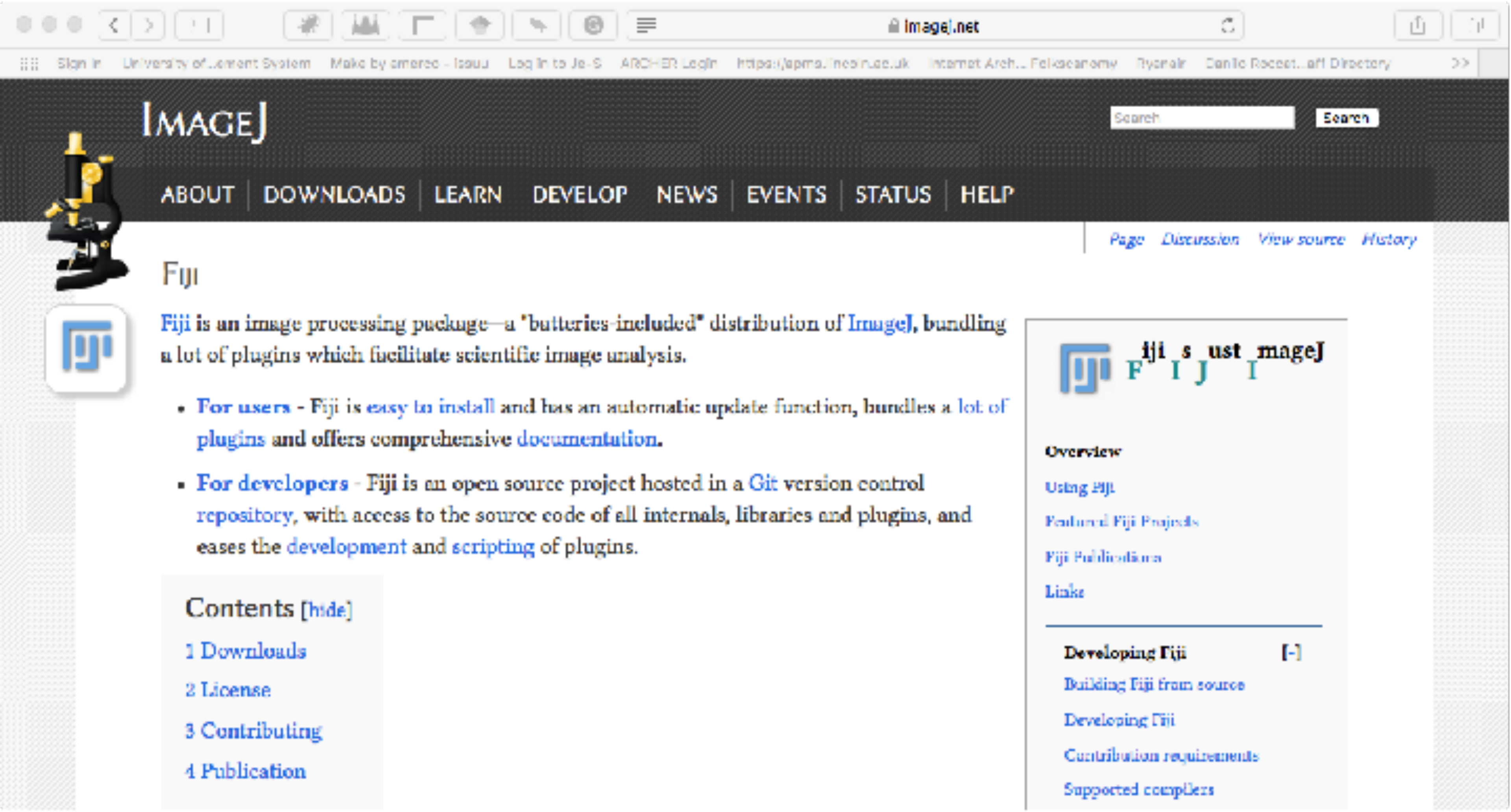


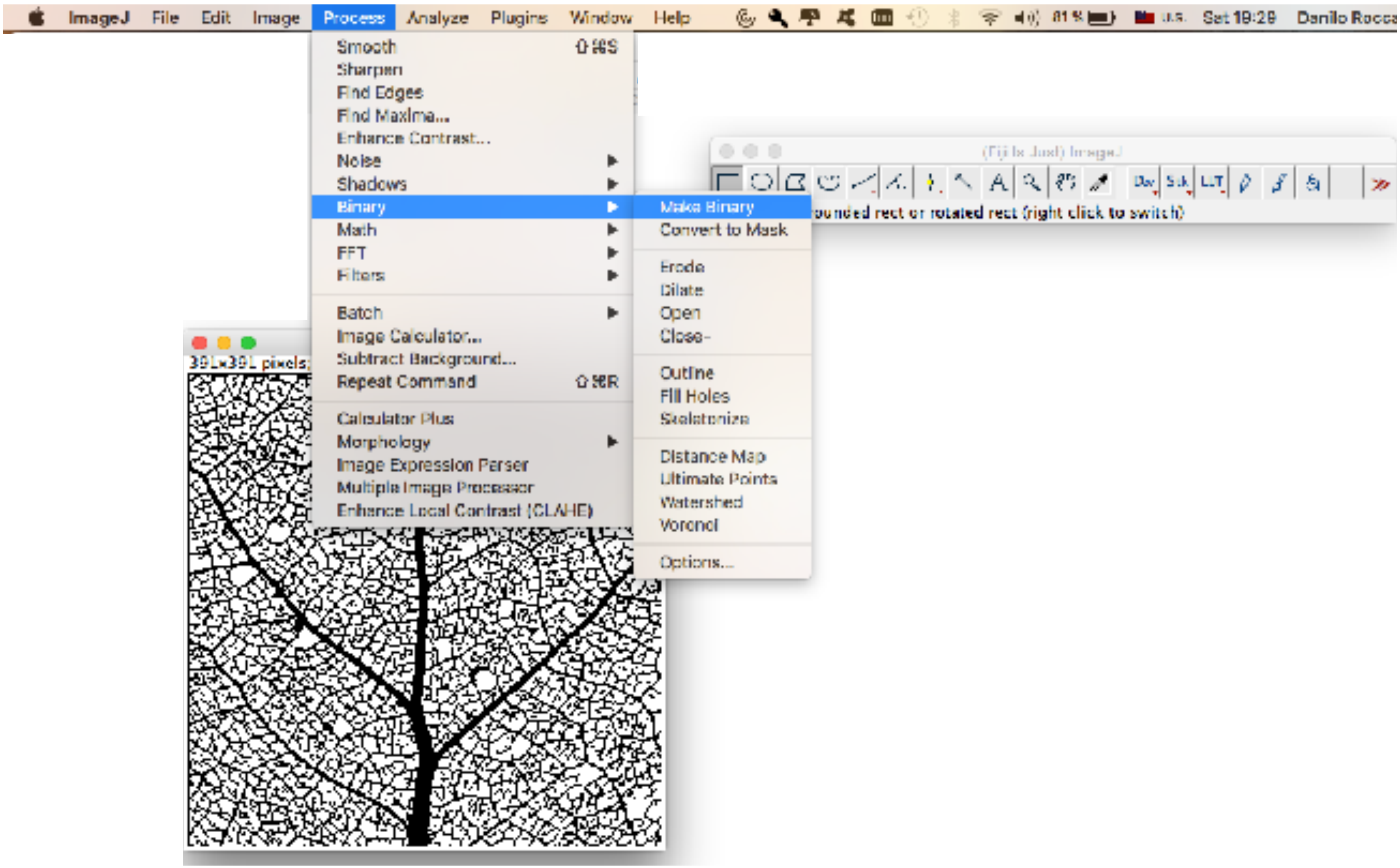


D=1.7668
10
8
6
log (count)
1
2
3
4
log (box size)

How to Calculate the Fractal Index using ImageJ

Download ImageJ/Fiji from: https://imagej.net/Fiji

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Batch
Image Calculator...
Subtract Background...
Repeat Command
Calculator Plus
Morphology
Image Expression Parser
Multiple Image Processor
Enhance Local Contrast (CLAHE)
Make Binary
Convert to Mask
Erode
Dilate
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Voronoi
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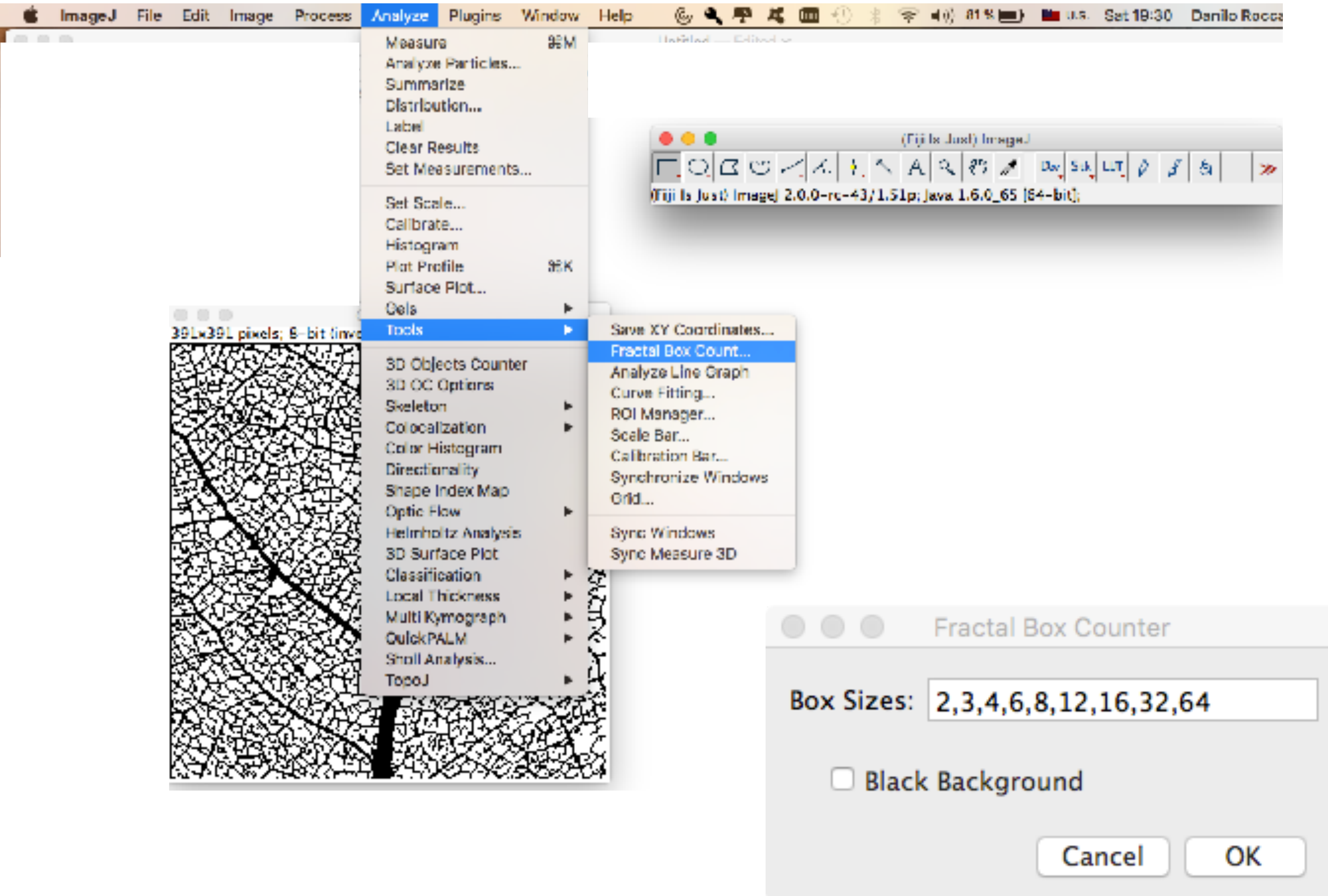
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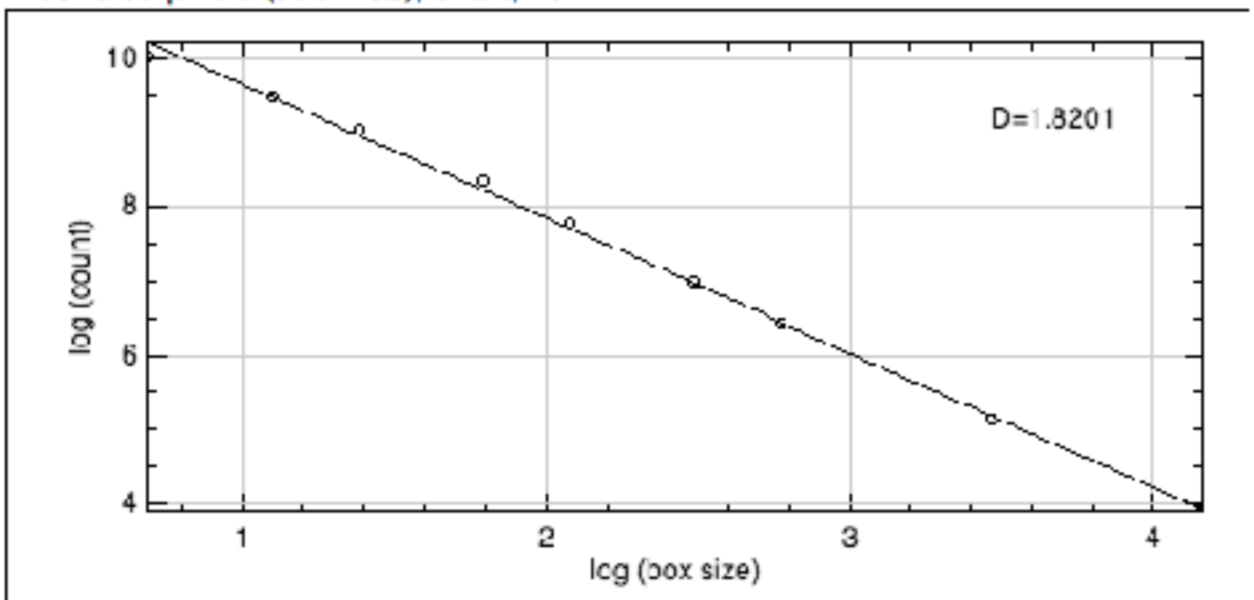
Summarize
Distribution...
Label
Clear Results
Set Measurements...
Set Scale...
Calibrate...
Histogram
Plot Profile
Surface Plot...
Gels
Tools
3D Objects Counter
3D OC Options
Skeleton
Colocalization
Color Histogram
Directionality
Shape Index Map
Optic Flow
Helmholtz Analysis
3D Surface Plot
Classification
Local Thickness
Multi Kymograph
QuickPALM
Sholl Analysis...
TopoJ
Save XY Coordinates...
Fractal Box Count...
Analyze Line Graph
Curve Fitting...
ROI Manager...
Scale Bar...
Calibration Bar...
Synchronize Windows
Grid...
Sync Windows
Sync Measure 3D
(Fiji Is Just) ImageJ
(Fiji Is Just) ImageJ 2.0.0-rc-43/1.51p; Java 1.6.0_65 [64-bit];
Fractal Box Counter
Box Sizes: 2,3,4,6,8,12,16,32,64
Black Background
Cancel
OK

Results

	Label	C2	C3	C4	C6	C8	C12	C16	C32	C64	D
1	Clipboard	23148	13345	8633	4269	2391	1089	625	169	49	1.820

Plot

4.08x8.09 pixels (530x255); 8-bit; 132K



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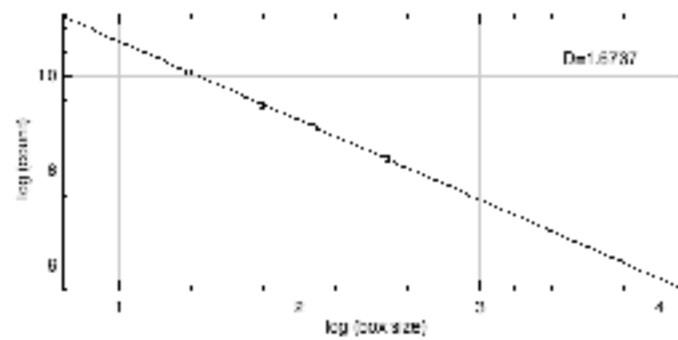
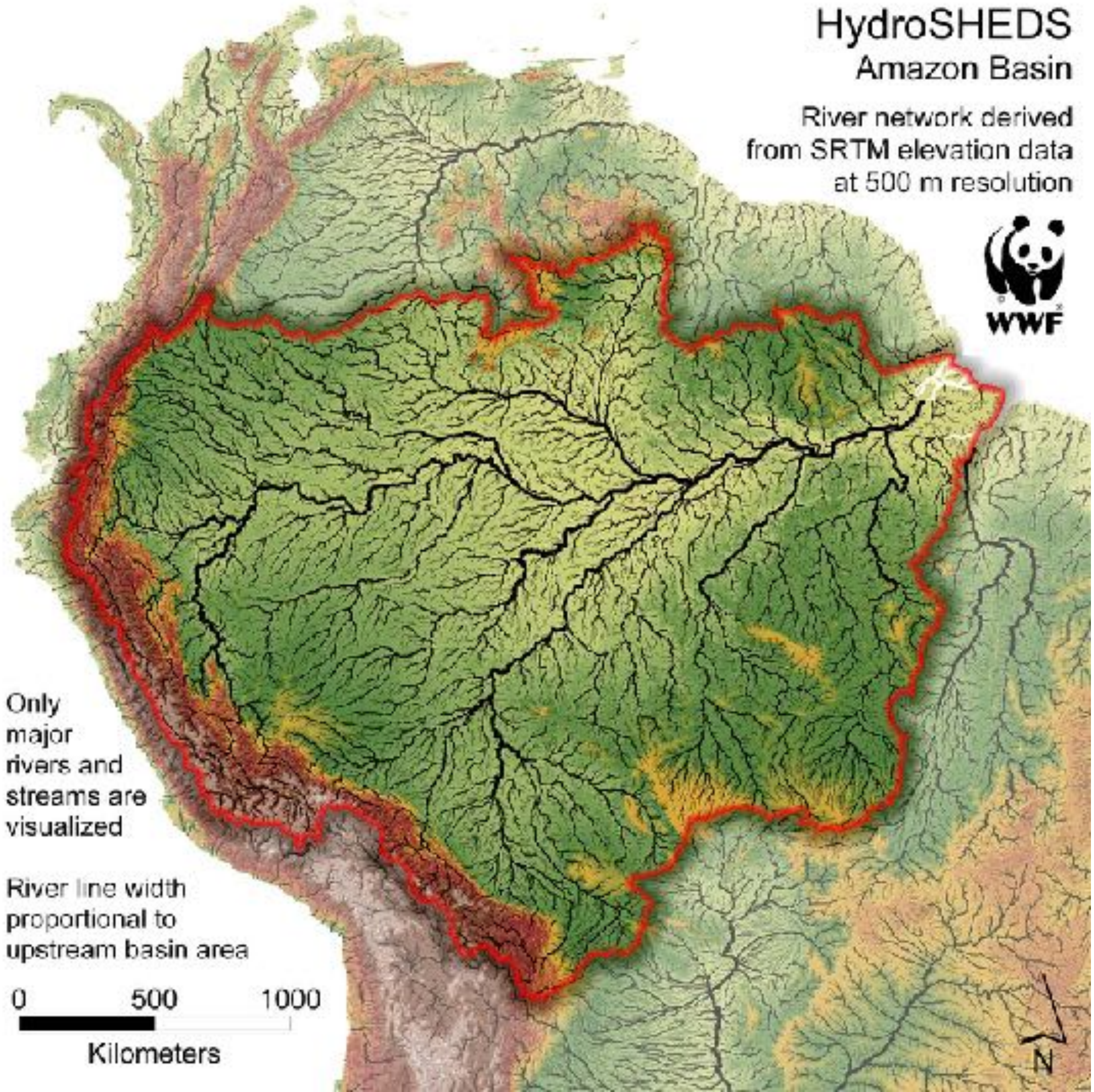
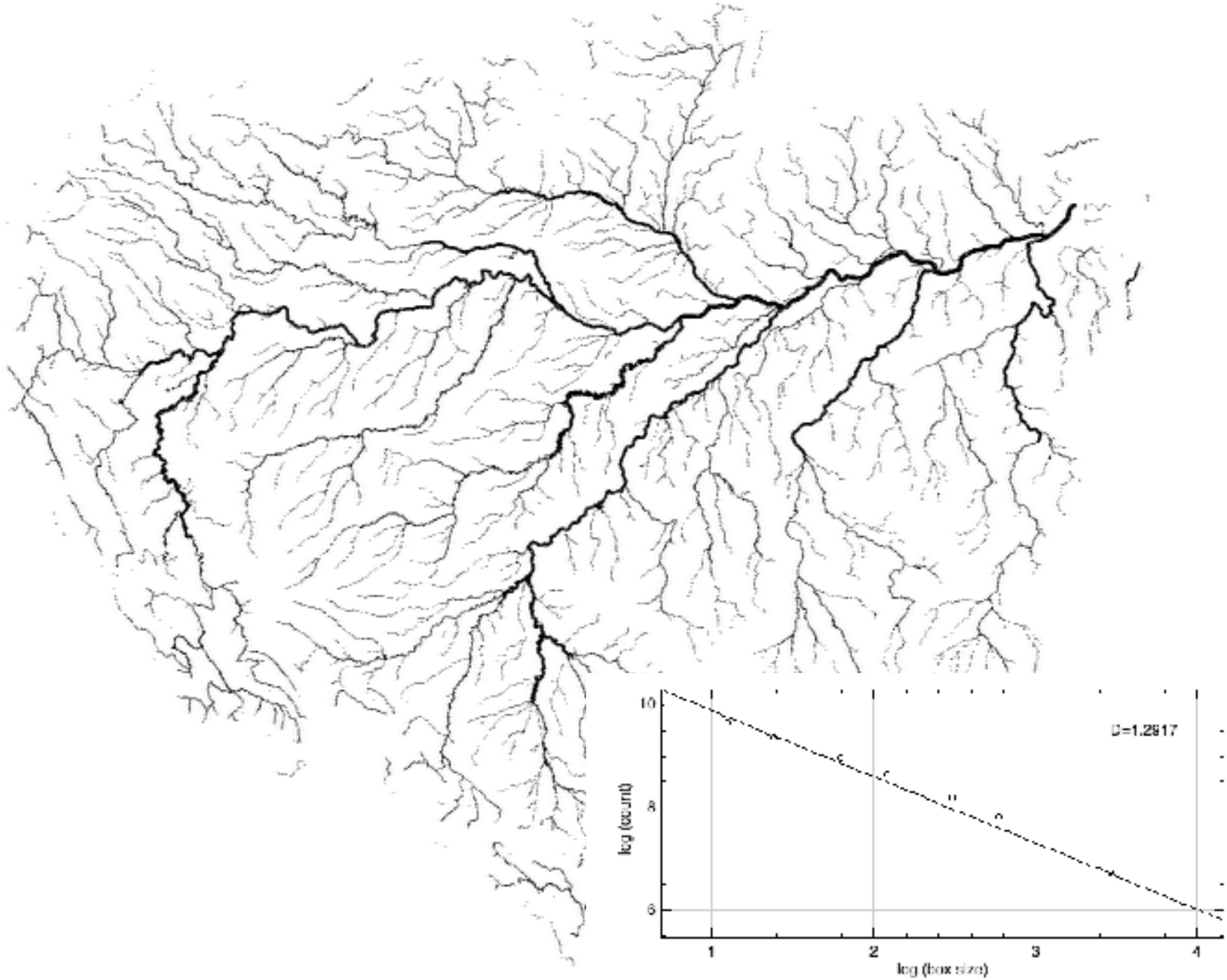
Dendritic deposit of
Calcium carbonate on the glass walls of a test tube




HydroSHEDS
Amazon Basin
River network derived
from SRTM elevation data
at 500 m resolution
WWF
Only
major
rivers and
streams are
visualized
River line width
proportional to
upstream basin area
0
500
1000
Kilometers
N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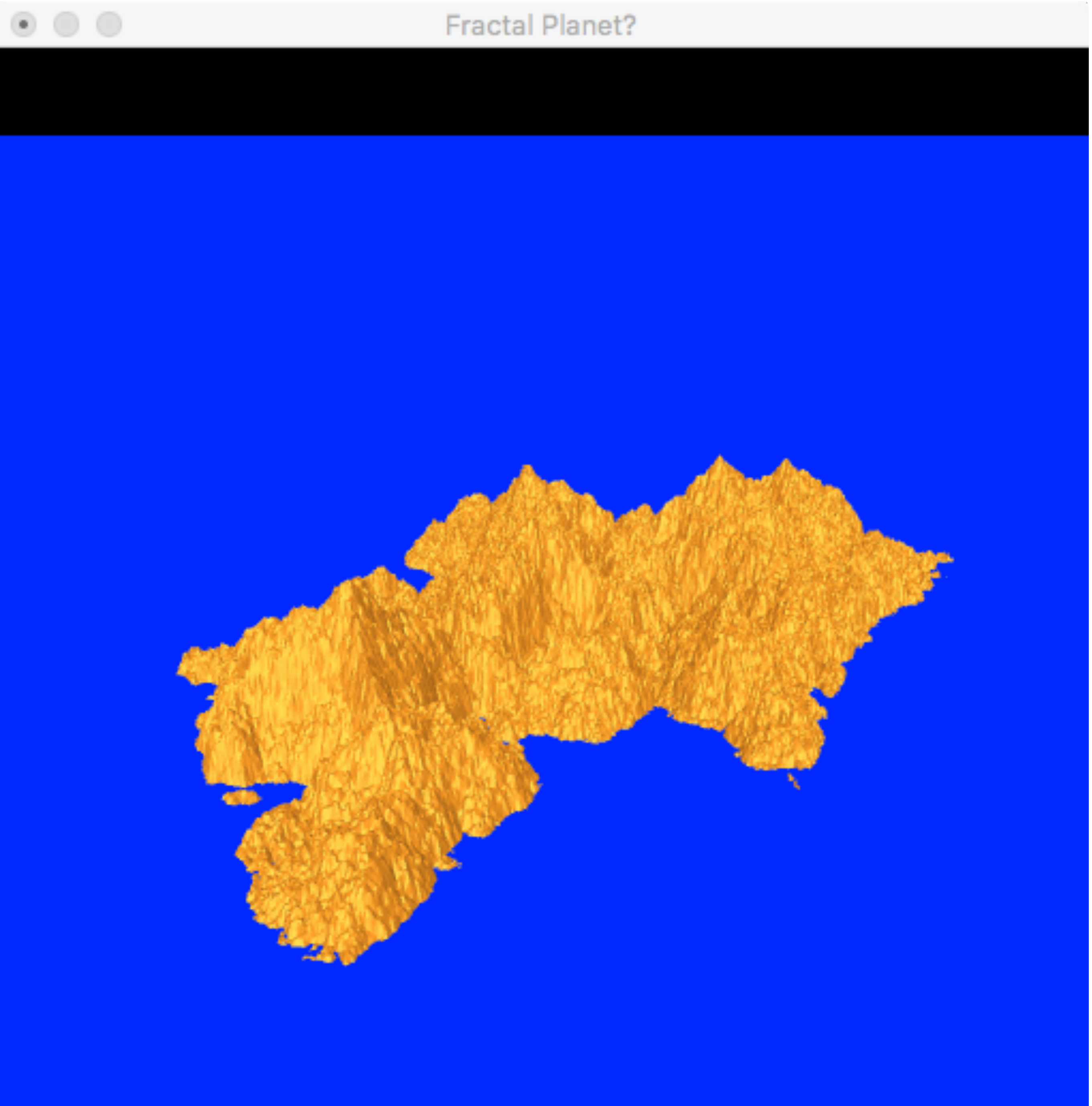
8
6
1
2
3
4

DIY A FRACTAL PLANET

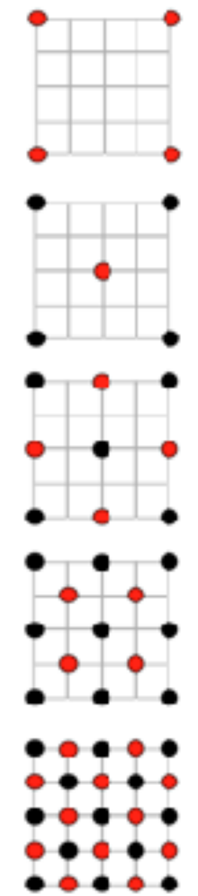
Star Trek II: The Wrath of Khan (1982)

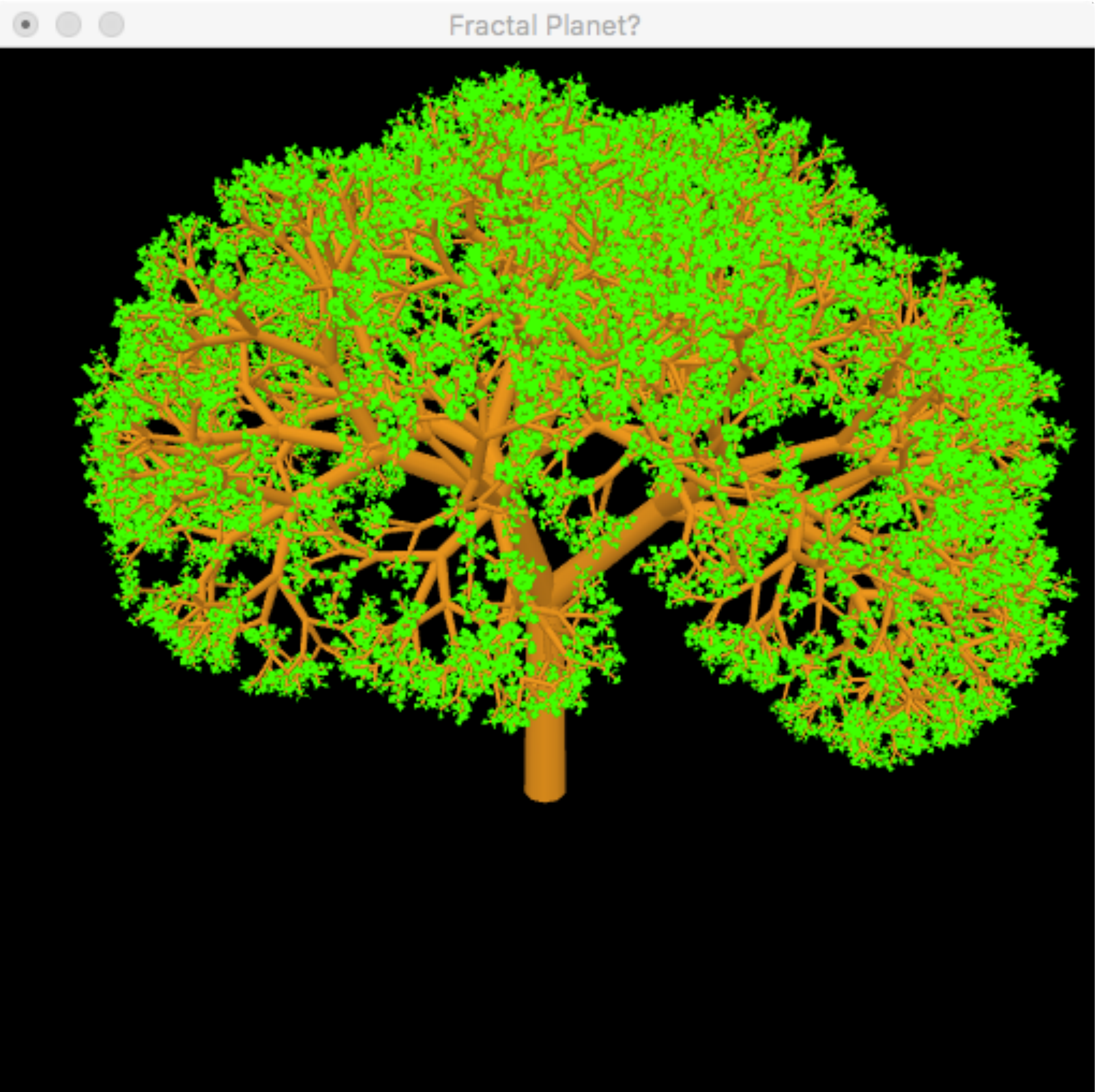


How to Build a Fractal Planet:

Islands with mountains

Square Diamond Mid-Point Displacement Algorithm



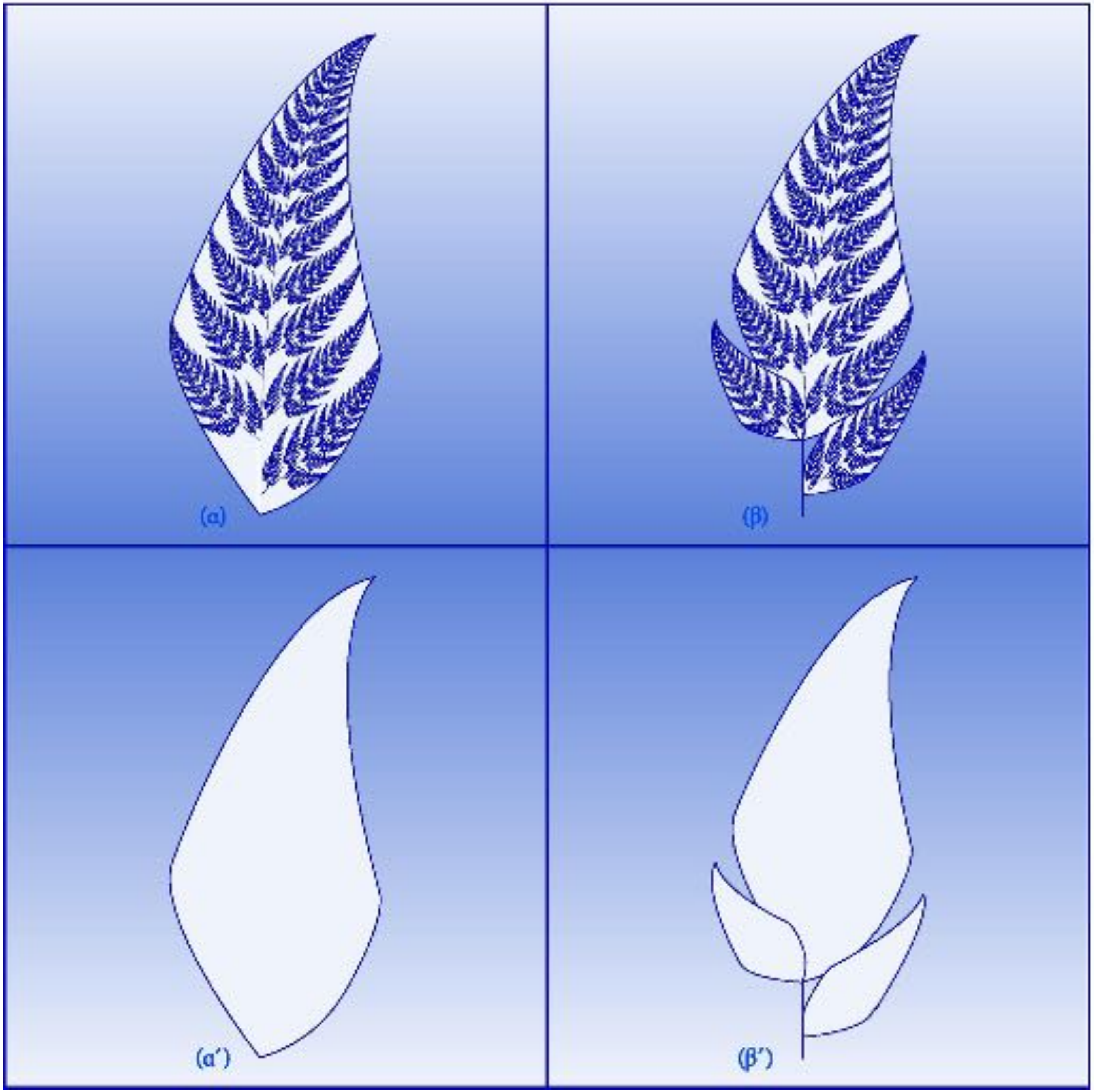


How to Build a Fractal Planet:

Vegetation

Lindenmayer system or L-System

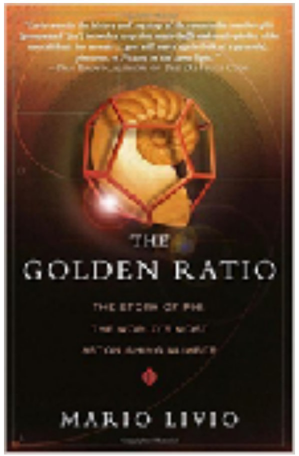
We can imitate nature by making pictures of real – looking plants like Barnsley's fern……



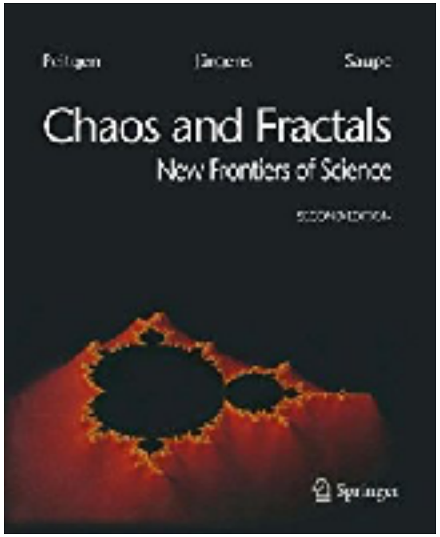
Starting with an initial pattern "close to the final product" can speed up the process very much..

Classic Readings

— Sir D'Arcy Wentworth Thompson
On Growth and Form (1917)



— Mario Livio
The golden ratio (2002)

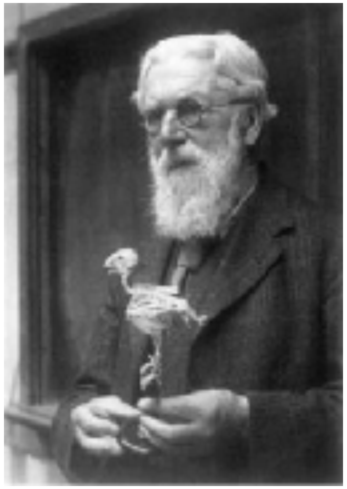


— Peitgen, Jürgens, Saupe
Chaos and Fractals
New Frontiers of Science (1992)

The Last Words …

For the harmony of the world is made manifest in Form and Number, and the heart and soul and all the poetry of Natural Philosophy are embodied in the concept of mathematical beauty.

— **Sir D'Arcy Wentworth Thompson**
On Growth and Form (1917), Epilogue, 778-9.



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