Network community detection under degree heterogeneity: spectral clustering with the random walk Laplacian

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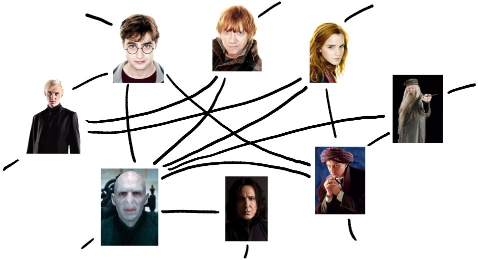
- Joint work with Patrick Rubin-Delanchy.

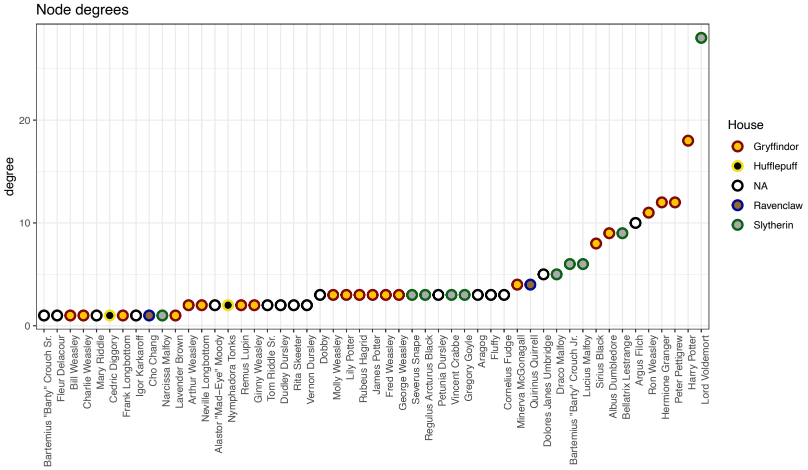
Running example

As a running example, we consider a network of 111 enmities between 51 characters from the Harry Potter book series.



Data available from `github.com/efekarakus/potter-network`.

Harry Potter enmity graph



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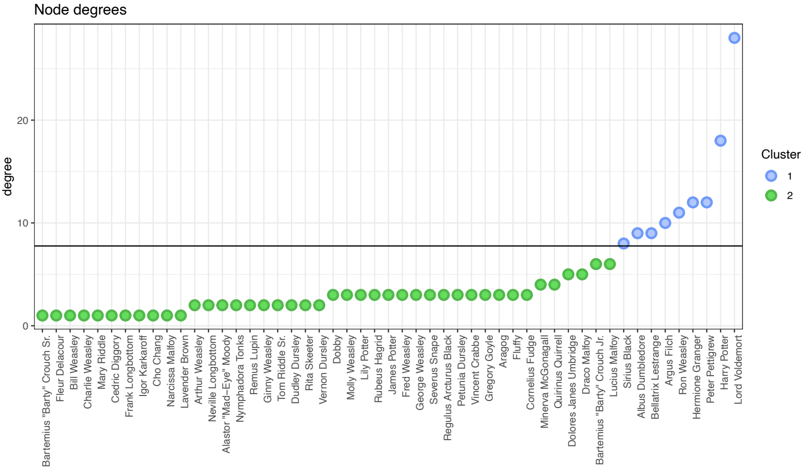

Spectral clustering algorithm

1. Form a matrix representation of the graph, e.g.
   - $\mathbf{A}$
     adjacency matrix,
   - $\mathbf{L}_{\text{un}} = \mathbf{D} - \mathbf{A}$
     unnormalised Laplacian matrix,
   - $\mathbf{L}_{\text{sym}} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$
     symmetric Laplacian matrix,
   - $\mathbf{L}_{\text{rw}} = \mathbf{D}^{-1}\mathbf{A}$
     random walk Laplacian matrix.

2. Compute it's d largest-in-magnitude eigenvalues $s_1, \ldots, s_d$ and eigenvectors $u_1, \ldots, u_d$.

3. Let $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_n)^\top := \mathbf{U}|\mathbf{S}|^{1/2}$ where
   - $\mathbf{U} := (u_1, \ldots, u_d)$,
   - $\mathbf{S} := \text{diag}(s_1, \ldots, s_d)$.

4. Cluster $\hat{X}_1, \ldots, \hat{X}_n$.

Stochastic block model (Holland et al., 1983)

Given

- a symmetric probability matrix $\mathbf{B} \in [0,1]^{K \times K}$,
- communities $z_1, \ldots, z_n \in \{1, \ldots, K\}$,

the adjacency matrix $\mathbf{A}$ follows a *stochastic block model* if

$$a_{ij} \overset{\text{ind.}}{\sim} \text{Bernoulli}(b_{z_i, z_j})$$

for $i < j$.

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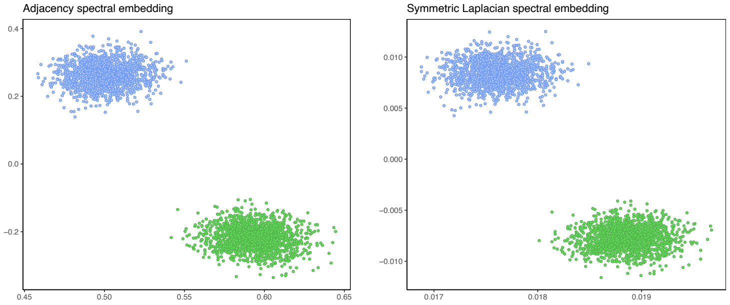

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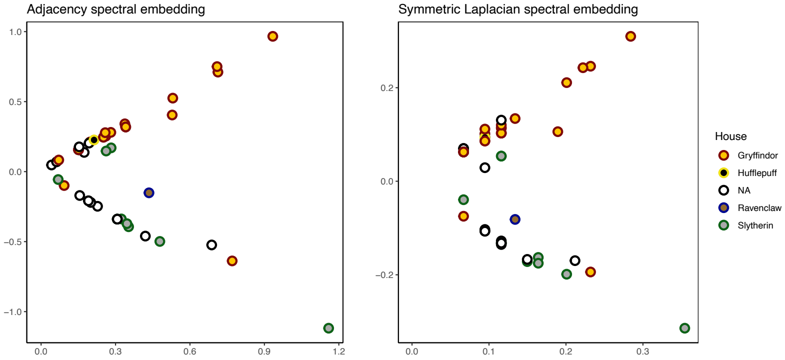

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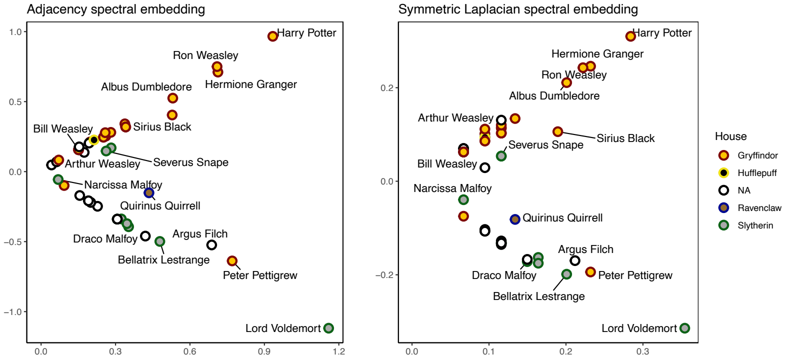

Degree-corrected stochastic block model (Karrer and Newman, 2011)

Given

- weights $w_1, \ldots, w_n \in [0, 1]$,
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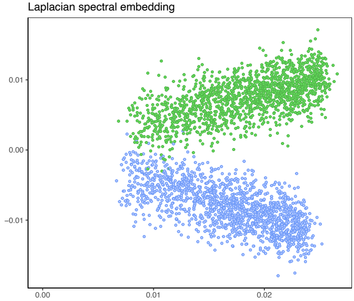

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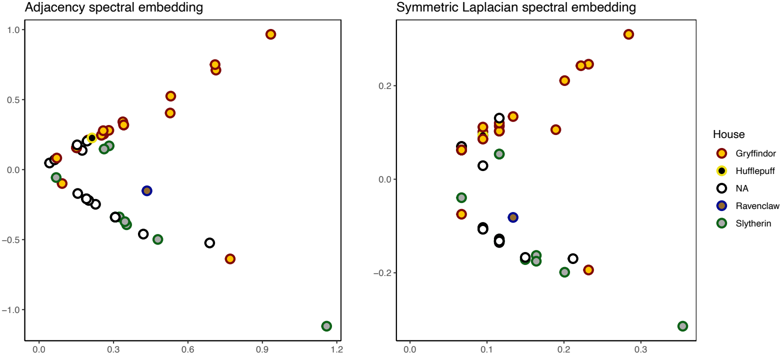

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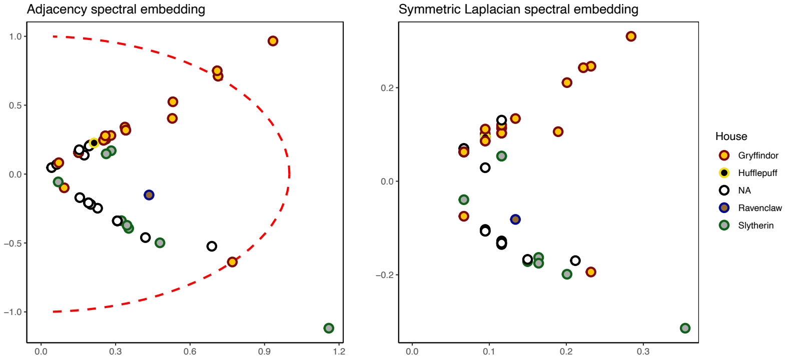

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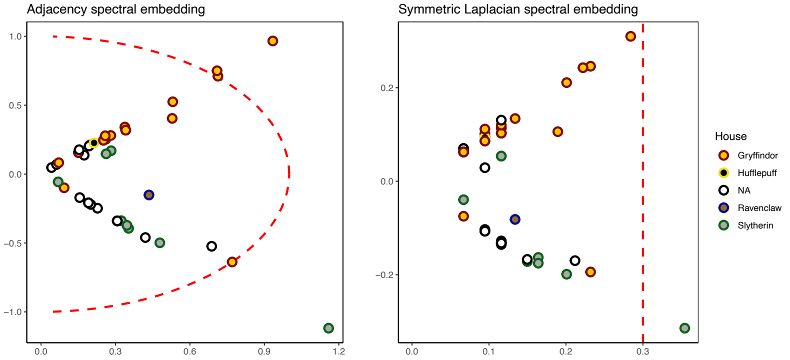

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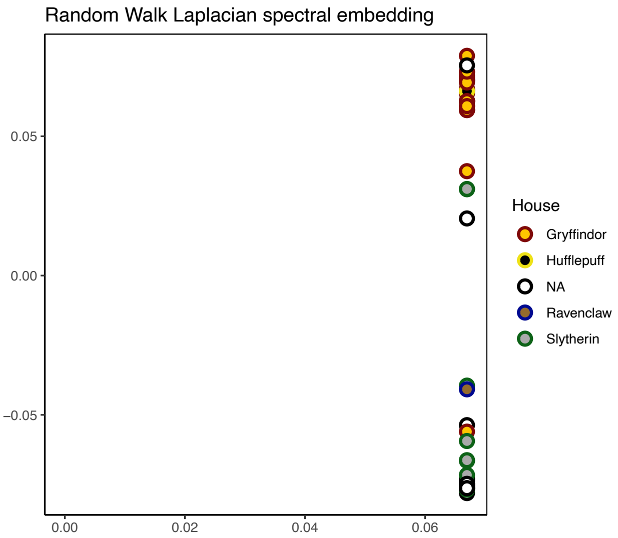

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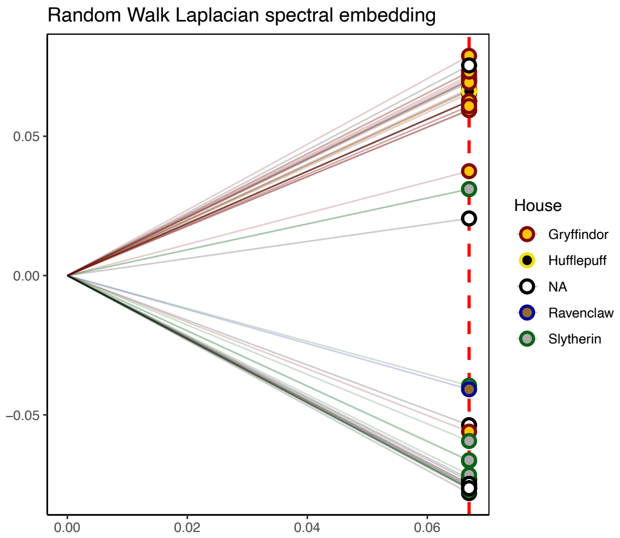

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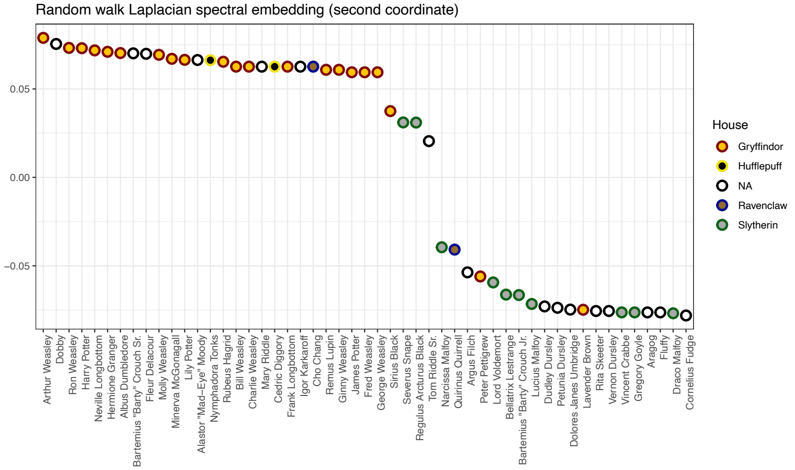

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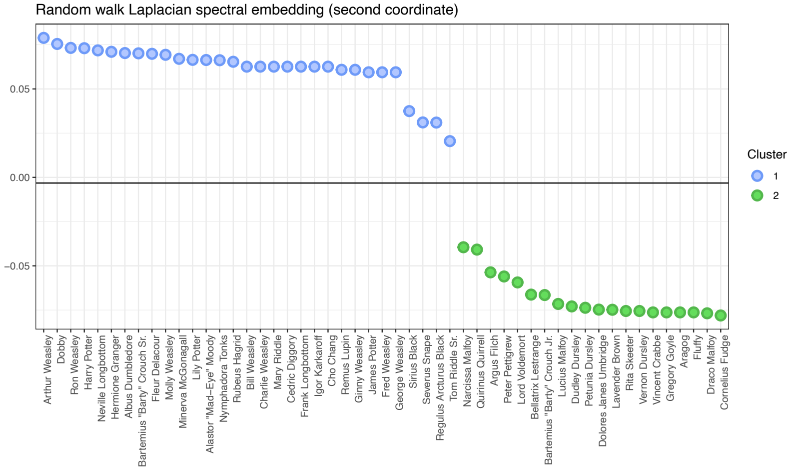

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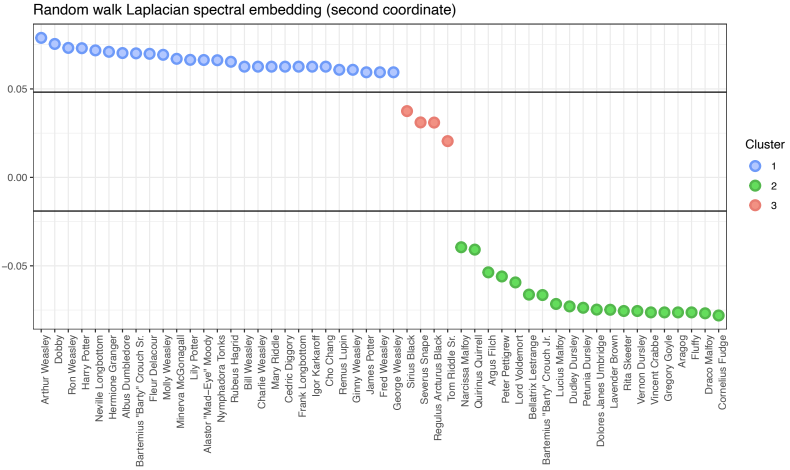

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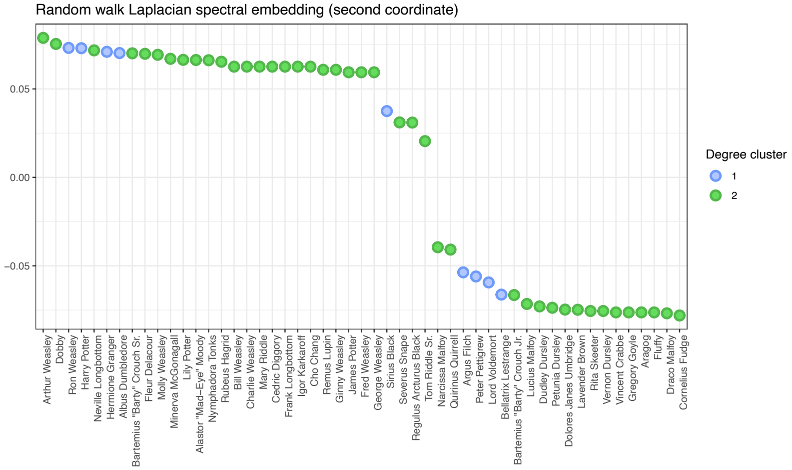

Random dot product graph (Young and Scheinerman, 2007)

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Let $v_1, \ldots, v_K \in \mathbb{R}^d$ be such that $b_{k\ell} = \langle v_k, v_\ell \rangle$, then

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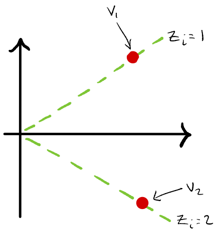

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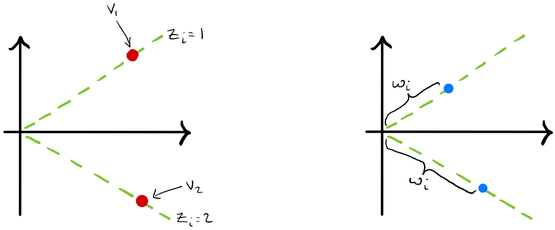


Spectral embedding as estimating a random dot product graph

Theorem (Rubin-Delanchy et al. (2017))

There exists an orthogonal matrix $\mathbf{Q}$ *such that with high probability...*

- *under adjacency spectral embedding*

$$\max_i \left\| \mathbf{Q}\hat{X}_i - X_i \right\| \to 0,$$

- *under symmetric Laplacian spectral embedding*

$$\max_i \left\| \mathbf{Q}\hat{X}_i - \frac{X_i}{\sqrt{t_i}} \right\| \to 0,$$

where $t_i = \sum_j \langle X_i, X_j \rangle$.

Theorem (Athreya et al. (2013); Tang and Priebe (2016))

The error distributions are asymptotically Gaussian.

New results

Theorem (Modell and Rubin-Delanchy (2021))

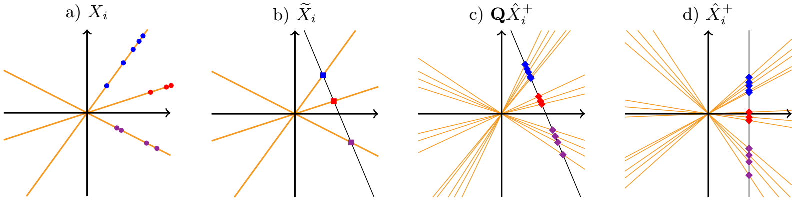
There exists an orthogonal matrix $\mathbf{Q}$ *such that under random walk Laplacian spectral embedding, with high probability,*

$$\max_i \left\| \mathbf{Q}\hat{X}_i - \tilde{X}_i \right\| \to 0,$$

where $\tilde{X}_i$ *is a projective plane representation of* X_i*, and*

$$n^{3/2}\left(\mathbf{Q}\hat{X}_i - \tilde{X}_i\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{\Sigma}(X_i))$$

where $\mathbf{\Sigma}(X_i)$ *has rank* $d-1$.



Projective plane representations of the latent positions

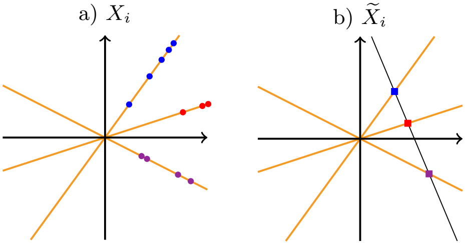
Lemma

Let $P_{\mathcal{H}}$ denote the projection onto the hyperplane

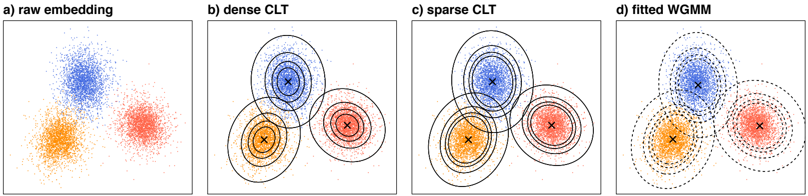
$$\mathcal{H} = \{x \in \mathbb{R}^d : (\sum_j X_j)^\top x = 1\},$$

then

$$\tilde{X}_i := P_{\mathcal{H}}(X_i) = \frac{X_i}{t_i}.$$



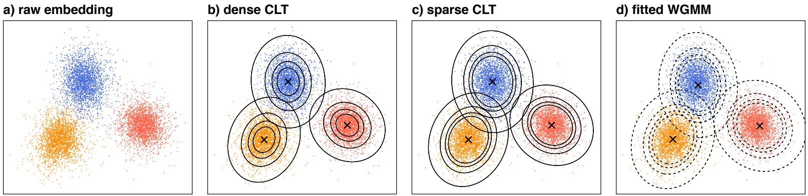
Central limit theorem



Demonstration of the central limit theorem for a degree-corrected stochastic block model with $n = 8000$ nodes and parameters

$$\mathbf{B} = \begin{pmatrix} 0.08 & 0.06 & 0.06 \\ 0.06 & 0.10 & 0.06 \\ 0.06 & 0.06 & 0.12 \end{pmatrix}, \quad w_1, \ldots, w_n \overset{\text{i.i.d.}}{\sim} \text{uniform}(0.25, 1), \quad \pi = (1/3, 1/3, 1/3).$$

Central limit theorem



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Remark

Under a sparse degree-corrected stochastic block model, $\mathbf{\Sigma}(X_i) \propto w_i^{-1}$.

References

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